JGP - Vol. 5, n. 1, 1988

Gluing of perverse sheaves and discrete series representation

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Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *We construct new abelian categories by gluing perverse sheaves using Fourier Transformations. These new categories are related to representation theory and conjecturally lead to new cohomological realizations of representations of reductive groups over finite fields.*

Résumé. *On construit de nouvelles catégories abéliennes par recollement de faisceaux pervers à l'aide de transformations de Fourier. Ces nouvelles catégories sont reliées à la théorie des représentations et fournissent conjecturalement de nouvelles réalisations cohomologiques des représentations des groupes réductifs sur les corps finis.*

INTRODUCTION

Let S be a scheme of finite type over an algebraically closed field and let $(S_i)_{i\in I}$ be a Zariski open covering of S ; we denote by

Key-Words: Cohomologie ℓ-adique, transformation de Fourier, théorie des représentations, groupes réductifs.

1980 MSC: 14 F, 14 G, 20 C.

$$\nu_{ij} : S_i \cap S_j \hookrightarrow S_i \quad (i, j \in I)$$

the inclusions.

The datum of an (ℓ-adic) perverse sheaf A on S is equivalent to the data of perverse sheaves A_i on S_i ($i \in I$) and isomorphisms

$$\nu_{ij}^* A_i \xrightarrow{\sim} \nu_{ji}^* A_j \qquad (i, j \in I)$$

satisfying the usual cocycle condition.

This can be reformulated in the following way. The functor ν_{ij}^* on perverse sheaves has two adjoint functors, one on the left ${}^p\nu_{ij,!}$ and one on the right ${}^p\nu_{ij,*}$ and we get by composition two functors

$${}^p\sigma_{ji,!} = {}^p\nu_{ji,!}\nu_{ij}^*$$

and

$${}^p\sigma_{ji,*} = {}^p\nu_{ji,*}\nu_{ij}^*$$

from the category of perverse sheaves on S_i to the category of perverse sheaves on S_j. We have obvious morphisms of functors

$$ {}^pc_{kji,!} : {}^p\sigma_{kj,!}{}^p\sigma_{ji,!} \to {}^p\sigma_{ki,!}$$

$$ {}^pc_{kji,*} : {}^p\sigma_{ki,*} \to {}^p\sigma_{kj,*}{}^p\sigma_{ji,*}$$

$(i, j, k \in I)$, satisfying the obvious associativity condition and a morphism of functors

$${}^pcan_{ji} : {}^p\sigma_{ji,!} \to {}^p\sigma_{ji,*}$$

induced by the canonical map

$${}^p\nu_{ji,!} \to {}^p\nu_{ji,*}.$$

Moreover, ${}^p\sigma_{ji,!}$ is left adjoint to ${}^p\sigma_{ij,*}$ and the adjunction exchanges ${}^pc_{kji,!}$ and ${}^pc_{ijk,*}$, ${}^pcan_{ji}$ and ${}^pcan_{ij}$. Then the category of perverse sheaves on S is equivalent to the category $\mathcal{A}$ of families

$$(A_i, \alpha_{ji})_{i,j\in I},$$

where A_i is a perverse sheaf on S_i and

$$\alpha_{ji} : {}^p\sigma_{ji,!}A_i \to A_j$$

is a map of perverse sheaves on S_j, such that the following axioms are satisfied:

1) for each $i, j, k \in I$, the following diagram of maps of perverse sheaves on S_k

$$\begin{array}{ccccc} {}^p\sigma_{kj,!}{}^p\sigma_{ji,!}A_i & \xrightarrow{{}^pc_{kji,!}(A_i)} & {}^p\sigma_{ki,!}A_i & \xrightarrow{\alpha_{ki}} & A_k \\ & {\scriptstyle {}^p\sigma_{kj,!}(\alpha_{ji})} \searrow & & \nearrow {\scriptstyle \alpha_{kj}} & \\ & & {}^p\sigma_{kj,!}A_j & & \end{array}$$

is commutative,

2) for each $i, j \in I$, if we denote by

$$\beta_{ji} : A_j \to {}^p\sigma_{ji,*}A_i$$

the map deduced by adjunction from α_{ij}, the following diagram of perverse sheaves on S_j

$$\begin{array}{ccc} {}^p\sigma_{ji,!}A_i & & \\ & \searrow {\scriptstyle \alpha_{ji}} & \\ {\scriptstyle {}^pcan_{ji}(A_i)} \downarrow & & A_j \\ & \swarrow {\scriptstyle \beta_{ji}} & \\ {}^p\sigma_{ji,*}A_i & & \end{array}$$

is commutative.

Now, if we have a family $(S_i)_{i \in I}$ of schemes of finite type over an algebraically closed field, functors

$${}^p\sigma_{ji,!}, {}^p\sigma_{ji,*}$$

from the category of perverse sheaves on S_i to the category of perverse sheaves on S_j $(i, j \in I)$ and morphisms of functors

$${}^pc_{kji,!} : {}^p\sigma_{kj,!}{}^p\sigma_{ji,!} \to {}^p\sigma_{ki,!}$$

$${}^pc_{kji,*} : {}^p\sigma_{ki,*} \to {}^p\sigma_{kj,*}{}^p\sigma_{ji,*}$$

and

$${}^pcan_{ji} : {}^p\sigma_{ji,!} \to {}^p\sigma_{ji,*}$$

having the same formal properties than the functors and morphisms of functors defined before, the category $\mathcal{A}$ makes sense and is easily seen to be abelian.

The first purpose of this paper is to give two interesting examples of such a category $\mathcal{A}$. In the first example, $I = \{1,2\}$, S_1 and S_2 are Zariski open subsets of a finite dimensional symplectic k-vector space V and the operators ${}^p\sigma_{ij,!}$ and ${}^p\sigma_{ij,*}$ are built using the Fourier-Deligne transformation from the category of perverse sheaves on V to itself. In the second example $I = W$ is the Weyl group of a simply connected, connected, semi-simple group over k endowed with a maximal torus T and a Borel subgroup B containing T, $S_i = G/U$ for each $i \in I$, where U is the unipotent radical of B and the operators ${}^p\sigma_{ij,!}$ are again built using Fourier-Deligne transformations.

Now, let S be a scheme of finite type over a finite field $\mathbf{F}_q$ and let σ be an automorphism of finite order of S. We can consider the abelian category of pairs (A, φ) where A is a perverse sheaf on $S \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q$ and

$$\varphi : \sigma^* Frob_q^* A \tilde{\to} A$$

is an isomorphism of perverse sheaves on $S \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q$. Let K be the Grothendieck group of this abelian category. We have a symmetric pairing

$$K \times K \to \overline{\mathbf{Q}}_\ell$$

defined in the following way. If (A_1, φ_1) and (A_2, φ_2) are in the above category, φ_1 and φ_2 induce an automorphism φ^i of the Yoneda higher extension group

$$Ext^i(A_1, DA_2) \qquad (i \in \mathbf{Z})$$

for the abelian category of perverse sheaves on $S \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q$, where D is the duality functor; thanks to Beilinson, these Yoneda higher extension groups are finite dimensional $\overline{\mathbf{Q}}_\ell$-vector spaces and the pairing maps $((A_1, \varphi_1), (A_2, \varphi_2))$ to

$$\sum_{i \in \mathbf{Z}} (-1)^i tr(\varphi^i).$$

This pairing is highly degenerate but we can consider the quotient of K by its kernel. It follows from Grothendieck trace formula that this quotient can be identified with the space of functions with values in $\overline{\mathbf{Q}}_\ell$ on the finite set

$$\{s \in S(\overline{\mathbf{F}}_q) | Frob_q(\sigma(s)) = s\}.$$

In particular, this quotient is a finite dimensional $\overline{\mathbf{Q}}_\ell$-vector space.

The second purpose of this paper is to generalize, at least conjecturally, the preceding discussion when we replace the abelian category of perverse sheaves on $S \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q$ by one of the abelian categories we have just constructed and σ^* by a more general operator on such a category. In this way, we expect to obtain new realizations of discrete series representations for reductive groups over finite fields.

1. THE BASIC GLUING CONSTRUCTION

(1.0) Let k be a field of characteristic $p > 0$ which is either finite or algebraically closed, let ℓ be a prime number not equal to p and let $\overline{\mathbf{Q}}_\ell$ be an algebraic closure of $\mathbf{Q}_\ell$.

Then for any scheme S of finite type over k, we have the derived category of $\overline{\mathbf{Q}}_\ell$-sheaves $D_c^b(S,\overline{\mathbf{Q}}_\ell)$ ([De](1.1)) with its autodual t-structure ([B-B-D] (4.0)); in particular, we have the abelian category $Perv(S,\overline{\mathbf{Q}}_\ell)$ of perverse sheaves on S and the cohomological functor ${}^p\mathcal{H}^0 : D_c^b(S,\overline{\mathbf{Q}}_\ell) \to Perv(S,\overline{\mathbf{Q}}_\ell)$. We will use freely this formalism.

(1.1) We will fix a nontrivial additive character

$$\psi : \mathbf{F}_p \to \overline{\mathbf{Q}}_\ell^\times$$

that is a p-th root of unity in $\overline{\mathbf{Q}}_\ell^\times$.

Then, on the additive group $\mathbf{G}_{a,\mathbf{F}_p}$ over $\mathbf{F}_p$, we have a smooth rank one $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{L}_\psi$ canonically attached to ψ, the so-called Artin-Schreier sheaf ([SGA $4\frac{1}{2}$] [Sommes trig.] (1.7)) and we will denote again by $\mathcal{L}_\psi$ the restriction of $\mathcal{L}_\psi$ to $\mathbf{G}_{a,S}$ for any scheme S over $\mathbf{F}_p$. If $f : T \to \mathbf{G}_{a,S}$ is a morphism of schemes over $\mathbf{F}_p$, we will denote by $\mathcal{L}_\psi(f)$ the pull-back of $\mathcal{L}_\psi$ (on $\mathbf{G}_{a,S}$) by f.

(1.2) Let Y be a smooth and connected scheme of finite type over k and let

$$\pi : V \to Y$$

a symplectic vector bundle of rank $2d \geq 2$ over Y: we have a symplectic pairing

$$\langle , \rangle : V \times_Y V \to \mathbf{G}_{a,k}.$$

The Fourier-Deligne transformation

$$\mathcal{F} = \mathcal{F}_\psi : D_c^b(V,\overline{\mathbf{Q}}_\ell) \to D_c^b(V,\overline{\mathbf{Q}}_\ell)$$

is defined by

$$\mathcal{F}(-) = Rpr_{1,!}(\mathcal{L}_\psi(\langle , \rangle) \otimes \tilde{pr}_2(-))$$

where

$$pr_1, pr_2 : V \times_Y V \to V$$

are the canonical projections and

$$\tilde{pr}_2(-) = pr_2^*(-)[2d](d) = pr_2^!(-)[-2d](-d)$$

(pr_2 is smooth of relative dimension $2d$). The following result is proved in [La] (1.2.2.1), (1.3.2.2) and (1.3.2.3):

THEOREM 1.2.1. (i) *$\mathcal{F}$ is an involution or $D^b_c(V,\overline{\mathbf{Q}}_\ell)$ ($\mathcal{F}^2 \simeq id$).*

(ii) *$\mathcal{F}$ commutes with duality; more precisely, if D is the duality functor on $D^b_c(V,\overline{\mathbf{Q}}_\ell)$,*

$$D\mathcal{F}_\psi \simeq \mathcal{F}_{\psi^{-1}} D.$$

(iii) *$\mathcal{F}$ is t-exact and induces an involution*

$$\mathcal{F} : Perv(V,\overline{\mathbf{Q}}_\ell) \to Perv(V,\overline{\mathbf{Q}}_\ell).$$ ∎

In fact, parts (ii) and (iii) of this theorem follow from the following statement ([Ka-La] (2.1.3)):

THEOREM 1.2.2. *The morphism of functors*

$$\mathcal{F}(-) \to R\,pr_{1,*}(\mathcal{L}_\psi(\langle,\rangle) \otimes \tilde{pr}_2(-))$$

induced by the canonical map

$$R\,pr_{1,!} \to R\,pr_{1,*}$$

is an isomorphism. ∎

(1.3) We can now begin the main construction of this chapter. Let

$$i : Z \hookrightarrow V$$

be a closed subset (for the Zariski topology) and let

$$j : X \hookrightarrow V$$

be the complementary open subset.

The Fourier-Deligne transformation $\mathcal{F}$ gives rise to two functors

$$F_! = F_{\psi,!} = j^*\mathcal{F}j_!$$

$$F_* = F_{\psi,*} = j^*\mathcal{F}Rj_*$$

from $D^b_c(X,\overline{\mathbf{Q}}_\ell)$ to itself with a canonical morphism of functors

$$can = can_\psi : F_! \to F_*$$

between them induced by the canonical map

$$j_! \to Rj_*.$$

LEMMA 1.3.1. (i) *If* D *is the duality functor on* $D_c^b(X,\overline{\mathbf{Q}}_\ell)$ *, then we have canonical isomorphisms of functors*

$$DF_{\psi,!} \simeq F_{\psi^{-1},*}D$$
$$DF_{\psi,*} \simeq F_{\psi^{-1},!}D;$$

moreover

$$D(can_\psi) = can_{\psi^{-1}}(D).$$

(ii) $(F_!, F_*)$ *is a pair of adjoint functors between* $D_c^b(X,\overline{\mathbf{Q}}_\ell)$ *and itself and the adjunction maps*

$$a = a_\psi : F_! F_* \to id$$
$$b = b_\psi : id \to F_* F_!$$

are induced by the adjunction maps

$$j_! j^* \to id$$
$$id \to Rj_* j^*$$

and the isomorphism of functors

$$\mathcal{F}^2 \simeq id;$$

moreover

$$D(a_\psi) = b_{\psi^{-1}}(D)$$
$$D(b_\psi) = a_{\psi^{-1}}(D).$$

(iii) $F_!$ (resp. F_*) *is* t*-exact on the right* (resp. *left).*

Proof. This lemma follows easily from (1.2.1), (1.2.2) and the properties of $(j_!, j^*, Rj_*)$ ([B-B-D] (1.4.2.1)). ∎

Taking the ${}^p\mathcal{H}^0$ of the functors F_1, F_*, we get functors

$${}^pF_! = {}^pF_{\psi,!} = j^*\mathcal{F}^p j_!$$
$${}^pF_* = {}^pF_{\psi,*} = j^*\mathcal{F}^p j_*$$

from $Perv(X,\overline{\mathbf{Q}}_\ell)$ into itself with a canonical morphism of functors

$${}^pcan = {}^pcan_\psi : {}^pF_! \to {}^pF_*$$

between them.

COROLLARY 1.3.2. (i) *We have canonical isomorphisms of functors*

$$D^pF_{\psi,!} \simeq {}^pF_{\psi^{-1},*}D$$
$$D^pF_{\psi,*} \simeq {}^pF_{\psi^{-1},!}D;$$

moreover

$$D({}^pcan_{\psi}) = {}^pcan_{\psi^{-1}}(D).$$

(ii) *For any* $A \in ob\ Perv(X,\overline{\mathbf{Q}}_\ell)$, *we have the following commutative diagram in* $D^b_c(X,\overline{\mathbf{Q}}_\ell)$

$$\begin{array}{ccc} F_!A & \overset{can}{\rightarrow} & F_*A \\ \downarrow & & \uparrow \\ {}^pF_!A & \overset{{}^pcan}{\rightarrow} & {}^pF_*A \end{array}$$

where the vertical arrows are induced by the canonical maps

$${}^p\tau_{\leq 0} \rightarrow id$$
$$id \rightarrow {}^p\tau_{\geq 0}.$$

(iii) $({}^pF_!, {}^pF_*)$ *is a pair of adjoint functors between* $Perv(X,\overline{\mathbf{Q}}_\ell)$ *and itself with adjunction maps*

$${}^pa = {}^pa_\psi : {}^pF_!{}^pF_* \rightarrow {}^p\mathcal{H}^0(F_!F_*) \overset{{}^p\mathcal{H}^0(a)}{\rightarrow} id$$
$${}^pb = {}^pb_\psi : id \overset{{}^p\mathcal{H}^0(b)}{\rightarrow} {}^p\mathcal{H}^0(F_*F_!) \rightarrow {}^pF_*{}^pF_!$$

(we have canonical morphisms of functors on $D^b_c(X,\overline{Q}_\ell)$

$$F_!{}^p\tau_{\leq 0} \overset{\sim}{\leftarrow} {}^p\tau_{\leq 0}F_!{}^p\tau_{\leq 0} \rightarrow {}^p\tau_{\leq 0}F_!$$
$${}^p\tau_{\geq 0}F_* \rightarrow {}^p\tau_{\geq 0}F_*{}^p\tau_{\geq 0} \overset{\sim}{\leftarrow} F_*{}^p\tau_{\geq 0});$$

moreover

$$D({}^pa_\psi) = {}^pb_{\psi^{-1}}(D)$$
$$D({}^ob_\psi) = {}^pa_{\psi^{-1}}(D).$$

(iv) ${}^pF_!$ *(resp.* ${}^pF_*$ *) is exact on the right (resp. left).*

Proof. Left to the reader. ∎

(1.4) Let A, A' be objects of $Perv(X, \overline{\mathbf{Q}}_\ell)$. To any pair of maps

$$ {}^pF_!A \xrightarrow{\alpha} A' \xrightarrow{\beta} {}^pF_*A $$

in $Perv(X, \overline{\mathbf{Q}}_\ell)$ we can associate by adjunction a pair of maps

$$ {}^pF_!A' \xrightarrow{\alpha'} A \xrightarrow{\beta'} {}^pF_*A' $$

in $Perv(X, \overline{\mathbf{Q}}_\ell)$ and conversely, starting from (α', β'), we get (α, β) back by adjunction again. We will denote simply by

$$ \left(A \frac{\alpha}{\beta} A' \right) $$

the data of A, A', α and β and by

$$ \left(A' \frac{\alpha'}{\beta'} A \right) $$

the data of A', A, α' and β' and we will denote by

$$ \left(A' \frac{\alpha'}{\beta'} A \right) = \mathcal{F} \left(A \frac{\alpha}{\beta} A' \right) $$

the fact that $(A' \frac{\alpha'}{\beta'} A)$ is obtained from $(A \frac{\alpha}{\beta} A')$ by adjunction.

DEFINITION 1.4.1. *We will say that* $(A \frac{\alpha}{\beta} A')$ *is admissible if the two following diagrams commute simultaneously*

$$ \begin{array}{ccc} {}^pF_!A & & \\ \downarrow {\scriptstyle {}^pcan(A)} & \searrow^{\alpha} & \\ & & A' \\ & \swarrow_{\beta} & \\ {}^pF_*A & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} {}^pF_!A' & & \\ \downarrow {\scriptstyle {}^pcan(A')} & \searrow^{\alpha'} & \\ & & A \\ & \swarrow_{\beta'} & \\ {}^pF_*A' & & \end{array} $$

$((A' \frac{\alpha'}{\beta'} A) = \mathcal{F}(A \frac{\alpha}{\beta} A'))$.

Let

$$ \mathcal{A} = \mathcal{A}_\psi $$

be the category with objects the admissible $(A \frac{\alpha}{\beta} A')$'s and with maps from $(A_1 \frac{\alpha_1}{\beta_1} A'_1)$ to $(A_2 \frac{\alpha_2}{\beta_2} A'_2)$ the pairs of maps

$$(u,u') = (A_1 \xrightarrow{u} A_2,\ A'_1 \xrightarrow{u'} A'_2)$$

such that the following diagram commutes

$$\begin{array}{ccccc}
{}^pF_!A_1 & \xrightarrow{\alpha_1} & A'_1 & \xrightarrow{\beta_1} & {}^pF_*A_1 \\
\downarrow{\scriptstyle {}^pF_!u} & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle {}^pF_*u} \\
{}^pF_!A_2 & \xrightarrow{\alpha_2} & A'_2 & \xrightarrow{\beta_2} & {}^pF_*A_2
\end{array}$$

Then $\mathcal{A}$ is a $\overline{\mathbf{Q}}_\ell$-linear category in a natural way and we have obvious $\overline{\mathbf{Q}}_\ell$-linear functors

$$\mathcal{F} = \mathcal{F}_\psi : \mathcal{A} \to \mathcal{A},\ (A \frac{\alpha}{\beta} A') \mapsto (A' \frac{\alpha'}{\beta'} A)$$

$$\mathcal{D} = \mathcal{D}_\psi : \mathcal{A}_\psi \to \mathcal{A}_{\psi^{-1}},\ (A \frac{\alpha}{\beta} A') \mapsto (DA \frac{D\beta}{D\alpha} DA')$$

such that

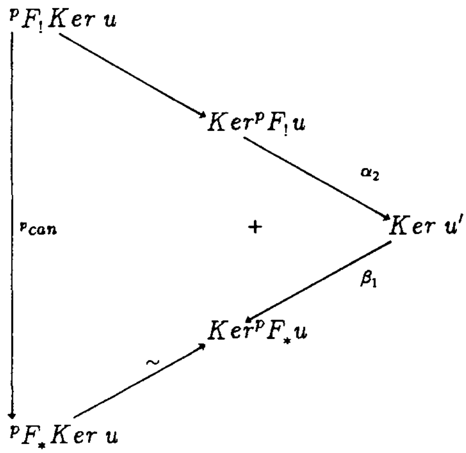
$$\mathcal{F}^2 \simeq id$$

$$\mathcal{D}_{\psi^{-1}}\mathcal{D}_\psi \simeq id$$

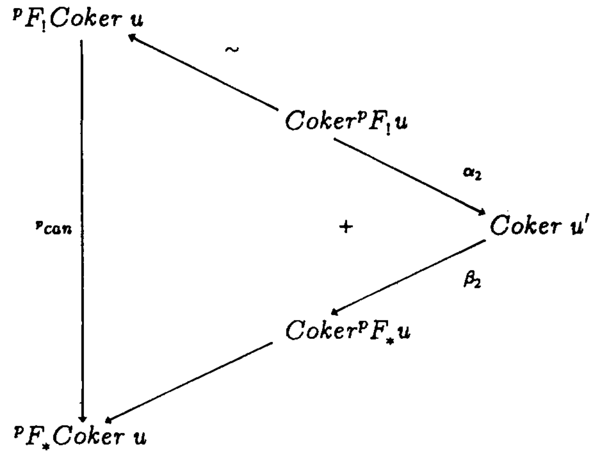
$$\mathcal{D}_\psi\mathcal{F}_\psi \simeq \mathcal{F}_{\psi^{-1}}\mathcal{D}_\psi .$$

LEMMA 1.4.2. *$\mathcal{A}$ is an abelian category and $\mathcal{F}$ and $\mathcal{D}$ are exact functors.*

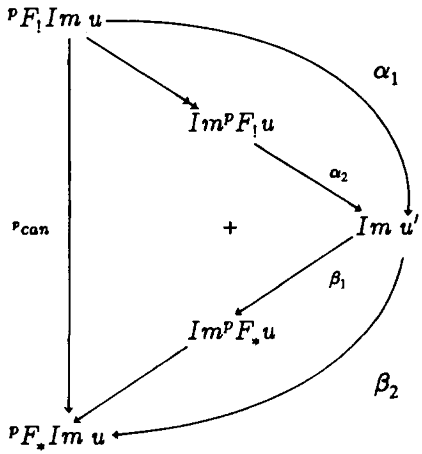
Proof. Let $(u,u') \in Hom_{\mathcal{A}}((A_1 \frac{\alpha_1}{\beta_1} A'_1),(A_2 \frac{\alpha_2}{\beta_2} A'_2))$, then



and



are kernel and cokernel of (u, u') respectively and



is at the same time the image and the coimage of (u, u'). This proves essentially that $\mathcal{A}$ is abelian. Moreover, it is clear that

$$\mathcal{F}\ Ker(u, u') = Ker(u', u)$$

$$\mathcal{F}\ Coker(u, u') = Coker(u', u)$$

and that

$$\mathcal{D}\ Ker(u, u') = Ker(Du, Du')$$

$$\mathcal{D}\ Coker(u, u') = Coker(Du, Du')$$

so $\mathcal{F}$ and $\mathcal{D}$ are exact. ∎

PROPOSITION 1.4.3. *The abelian category $\mathcal{A}$ is artinian and noetherian. The simple objects of $\mathcal{A}$ are of one of the following three types:*

(1) $(A - O)$ *where A is a simple object in $Perv(X,\overline{\mathbf{Q}}_\ell)$ such that ${}^p can : {}^pF_!A \to {}^pF_*A$ is identically zero;*
(2) $(O - A')$ *where A' is a simple object in $Perv(X,\overline{\mathbf{Q}}_\ell)$ such that ${}^p can : {}^pF_!A' \to {}^pF_*A'$ is identically zero;*
(3) $(A \overset{\alpha}{\underset{\beta}{-}} A')$ *where A and A' are simple objects in $Perv(X,\overline{\mathbf{Q}}_\ell)$ and where*

$$ {}^pF_!A \overset{\alpha}{\twoheadrightarrow} A' \overset{\beta}{\hookrightarrow} {}^pF_*A $$

is the canonical factorization of ${}^p can$ through its image (α is an epimorphism and β is a monomorphism in $Perv(X,\overline{\mathbf{Q}}_\ell)$ which is equivalent to α and β non zero).

Moreover $\mathcal{F}$ exchanges the types (1) and (2) and fixes the type (3) and $\mathcal{D}$ fixes all the types.

Proof. The first assertion follows immediately from the similar one for $Perv(X,\overline{\mathbf{Q}}_\ell)$ ([B-B-D](4.3.1)(i)).

Now, if $(A \overset{\alpha}{\underset{\beta}{-}} A')$ is simple, then A and A' are either zero or simple objects: for example, if A has a non trivial quotient $A \overset{u}{\twoheadrightarrow} \overline{A}$ in $Perv(X,\overline{\mathbf{Q}}_\ell)$, let $A' \overset{u'}{\twoheadrightarrow} \overline{A}'$ be the quotient of A' by $\alpha(Ker\,{}^pF_!u)$ and let $\overline{\alpha} : {}^pF_!\overline{A} \to \overline{A}'$ and $\overline{\beta} : \overline{A}' \to {}^pF_*\overline{A}$ be the maps induced by α and β respectively (we have $\beta \circ \alpha(Ker\,{}^pF_!u) = {}^p can(Ker\,{}^pF_!u) \subset {}^pF_*Ker\, u$) ; then it is easy to see that $(\overline{A} \overset{\overline{\alpha}}{\underset{\overline{\beta}}{=}} \overline{A}')$ is admissible and that (u,u') is a non trivial quotient of $(A \overset{\alpha}{\underset{\beta}{-}} A')$ in $\mathcal{A}$.

Next we remark that for any $(A \overset{\alpha}{\underset{\beta}{-}} A')$ in $\mathcal{A}$ with A and A' simple, α is either zero or an epimorphism and β is either zero or a monomorphism.

Finally, if $(A \overset{\alpha}{\underset{\beta}{-}} A') \in ob\,\mathcal{A}$ with A and A' simple has α (resp. β) equal to zero, then $(A - O) \hookrightarrow (A \overset{\alpha}{\underset{\beta}{-}} A')$ (resp. $(A \overset{\alpha}{\underset{\beta}{-}} A') \twoheadrightarrow (A - O)$) is a non trivial subobject (resp. quotient) of $(A \overset{\alpha}{\underset{\beta}{-}} A')$ and $(A \overset{\alpha}{\underset{\beta}{-}} A')$ cannot be simple. The proposition follows. ∎

(1.5) The abelian categories $\mathcal{A}$ and $Perv(V,\overline{\mathbf{Q}}_\ell)$ are related by a sequence of three adjoint functors

$$ (\nu_!, \nu^*, \nu_*) . $$

More precisely, for any object B and map v in $Perv(V, \overline{\mathbf{Q}}_\ell)$, let

$$\nu^* B = (j^* B \underset{\beta}{\overset{\alpha}{-}} j^* \mathcal{F} B)$$

where α and β are induced by the adjunction maps ${}^p j_! j^* \to id$ and $id \to {}^p j_* j^*$ respectively and let

$$\nu^*(v) = (j^*(v), j^* \mathcal{F}(v)).$$

Then it is easy to see that $\nu^* B$ is admissible and that $\nu^*(v)$ is a map in $\mathcal{A}$, so we have defined a functor

$$\nu^* = \nu^*_\psi : Perv(V, \overline{\mathbf{Q}}_\ell) \to \mathcal{A}.$$

It is obvious from (1.2.1) that ν^* is an exact functor, that

$$\mathcal{F} \nu^* \simeq \nu^* \mathcal{F}_V$$

and that

$$\mathcal{D}_\psi \nu^*_\psi \simeq \nu^*_{\psi^{-1}} D_V$$

where we denote by $\mathcal{F}_V$ and D_V the Fourier-Deligne transformation and the duality functor on $Perv(V, \overline{\mathbf{Q}}_\ell)$ respectively.

LEMMA 1.5.1. *The functor*

$$\nu^* : Perv(V, \overline{\mathbf{Q}}_\ell) \to \mathcal{A}$$

admits a left adjoint functor

$$\nu_! = \nu_{\psi,!} : \mathcal{A} \to Perv(V, \overline{\mathbf{Q}}_\ell)$$

and a right adjoint functor

$$\nu_* = \nu_{\psi,*} : \mathcal{A} \to Perv(V, \overline{\mathbf{Q}}_\ell).$$

Moreover:

(i) $\nu_!$ *is right exact and* ν_* *is left exact;*

(ii) *we have*

$$\mathcal{F}_V \nu_! \simeq \nu_! \mathcal{F}$$
$$\mathcal{F}_V \nu_* \simeq \nu_* \mathcal{F}$$

and

$$D_V \nu_{\psi,!} \simeq \nu_{\psi^{-1},*} \mathcal{D}_\psi$$
$$D_V \nu_{\psi,*} \simeq \nu_{\psi^{-1},!} \mathcal{D}_\psi;$$

(iii) *there exists a canonical morphism of functors*

$$\nu_! \to \nu_*$$

such that the following diagrams commute

$$\begin{array}{ccc} id & \to & id \\ \uparrow & & \downarrow \\ \nu_! \nu^* & \to & \nu_* \nu^* \end{array} \quad \text{and} \quad \begin{array}{ccc} id & \to & id \\ \downarrow & & \uparrow \\ \nu^* \nu_! & \to & \nu^* \nu_* \end{array}$$

(the two top horizontal arrows are identities, the two bottom horizontal arrows are induced by $\nu_! \to \nu_*$ and the four vertical arrows are adjunction maps) and

$$\mathcal{F}_V(\nu_! \to \nu_*) = (\nu_! \to \nu_*)\mathcal{F}$$

and

$$D_V(\nu_{\psi,!} \to \nu_{\psi,*}) = (\nu_{\psi^{-1},!} \to \nu_{\psi^{-1},*})\mathcal{D}_\psi$$

for this canonical map.

Proof. We will just give the formulas for $\nu_!$, ν_* and $\nu_! \to \nu_*$ and leave the details to the reader.

For any $(A \underset{\beta}{\overset{\alpha}{-}} A') \in ob\,\mathcal{A}$ with $\mathcal{F}(A \underset{\beta}{\overset{\alpha}{-}} A') = (A' \underset{\beta'}{\overset{\alpha'}{-}} A)$, we have functorial maps in $Perv(V, \overline{Q}_\ell)$

$$\mathcal{F}^p j_! j^* \mathcal{F}^p j_! A \oplus {}^p j_! j^* \mathcal{F}^p j_! A' \to {}^p j_! A \oplus \mathcal{F}^p j_! A'$$
$${}^p j_* A \oplus \mathcal{F}^p j_* A' \to \mathcal{F}^p j_* j^* \mathcal{F}^p j_* A \oplus {}^p j_* j^* \mathcal{F}^p j_* A'$$

and

$$^{p}j_{!}A \oplus \mathcal{F}^{p}j_{!}A' \to {}^{p}j_{*}A \oplus \mathcal{F}^{p}j_{*}A'$$

given by the following matrices

$$N_{!} = \begin{pmatrix} (\mathcal{F}^{p}j_{!}j^{*}\mathcal{F} \to id)({}^{p}j_{!}A) & {}^{p}j_{!}\alpha' \\ -\mathcal{F}^{p}j_{!}\alpha & -({}^{p}j_{!}j^{*} \to id)(\mathcal{F}^{p}j_{!}A') \end{pmatrix}$$

$$N_{*} = \begin{pmatrix} (id \to \mathcal{F}^{p}j_{*}j^{*}\mathcal{F})({}^{p}j_{*}A) & \mathcal{F}^{p}j_{*}\beta \\ -{}^{p}j_{*}\beta' & -(id \to {}^{p}j_{*}j^{*})(\mathcal{F}^{p}j_{*}A') \end{pmatrix}$$

and

$$\Delta = \begin{pmatrix} ({}^{p}j_{!} \to {}^{p}j_{*})(A) & 0 \\ 0 & \mathcal{F}({}^{p}j_{!} \to {}^{p}j_{*})(A') \end{pmatrix}$$

where ${}^{p}j_{!} \to {}^{p}j_{*}$ is the canonical map. Moreover, it is easy to see that the admissibility of $(A \underset{\beta}{\overset{\alpha}{-}} A')$ implies that the product of matrices

$$N_{*}\Delta N_{!}$$

is identically zero, so that Δ admits a functorial canonical factorization

$$^{p}j_{!}A \oplus \mathcal{F}^{p}j_{!}A' \twoheadrightarrow Coker(N_{!}) \to Ker(N_{*}) \hookrightarrow {}^{p}j_{*}A \oplus \mathcal{F}^{p}j_{*}A'.$$

Now, we can take

$$\nu_{!}(A \underset{\beta}{\overset{\alpha}{-}} A') = Coker(N_{!})$$

$$\nu_{*}(A \underset{\beta}{\overset{\alpha}{-}} A') = Ker(N_{*})$$

and

$$\nu_{!}(A \underset{\beta}{\overset{\alpha}{-}} A') \to \nu_{*}(A \underset{\beta}{\overset{\alpha}{-}} A')$$

induced by Δ . ■

LEMMA 1.5.2. *The adjunction maps*

$$\nu^{*}\nu_{*} \to id$$

$$id \to \nu^{*}\nu_{!}$$

are isomorphisms.

Proof. Using duality, we see immediately that it is enough to prove that $id \to \nu^*\nu_!$ is an isomorphism. Let us put

$$\nu^*\nu_!(A \frac{\alpha}{\beta} A') = (\tilde{A} \frac{\tilde{\alpha}}{\tilde{\beta}} \tilde{A}')$$

and denote by

$$(A \xrightarrow{u} \tilde{A},\ A' \xrightarrow{u'} \tilde{A}')$$

the adjunction map. We have to prove that u and u' are isomorphisms. Using Fourier-Deligne transformation, we see immediately that it is enough to prove that u is an isomorphism.

But, by definition, $\tilde{A}$ is the cokernel of the following map in $Perv(X, \overline{\mathbf{Q}}_\ell)$

$${}^pF_!{}^pF_!A \oplus {}^pF_!A' \to A \oplus {}^pF_!A'$$

with matrix

$$\begin{pmatrix} ({}^pF_!{}^pF_! \to id)(A) & \alpha' \\ -{}^pF_!\alpha & -id \end{pmatrix}$$

and it is clear that the inclusion

$$A = A \oplus 0 \hookrightarrow A \oplus {}^pF_!A'$$

identifies $\tilde{A}$ with the cokernel of

$$({}^pF_!{}^pF_! \to id)(A) - \alpha' \circ ({}^pF_!\alpha) : {}^pF_!{}^pF_!A \to A.$$

As this last map is identically zero by admissibilty of $(A \frac{\alpha}{\beta} A')$, the lemma follows. ∎

COROLLARY 1.5.3. *The following conditions are equivalent:*

(i) $\nu^* : Perv(V, \overline{\mathbf{Q}}_\ell) \to \mathcal{A}$ *is an equivalence of categories;*

(ii) *for any* $B \in ob\, Perv(V, \overline{Q}_\ell)$, $B = 0$ *if and only if* $\nu^*B = 0$;

(iii) *the adjunction maps*

$$\nu_!\nu^* \to id$$

$$id \to \nu_*\nu^*$$

and the canonical map

$$\nu_! \to \nu_*$$

are isomorphisms.

Proof. Left to the reader. ∎

Now we can state and prove our first main result:

THEOREM 1.5.4. *Let us assume that, for each $y \in Y$, $\dim Z_y < d$, where Z_y is the fiber at y of the canonical projection $Z \to Y$. Then*

$$\nu^* : Perv(V, \overline{\mathbf{Q}}_\ell) \to \mathcal{A}$$

is an equivalence of categories.

Proof. We will check the condition Corollary (1.5.3) (ii). Let $B \in ob\ Perv(V, \overline{\mathbf{Q}}_\ell)$ such that $j^*B = 0$ and $j^*\mathcal{F}B = 0$. Then $B = i_*C$ and $\mathcal{F}B = i_*C'$ for perverse sheaves C and C' on Z. Moreover

$$C' = R\,\overline{pr}_{2!}(\mathcal{L}_\psi(\overline{\langle,\rangle}) \otimes \overline{pr}_1^*C)[2d](d)$$

where we have denoted by

$$\overline{pr}_1, \overline{pr}_2 : Z \times_Y Z \to Z$$

the two canonical projections and by

$$\overline{\langle,\rangle} : Z \times_Y Z \to \mathbf{A}_Y^1$$

the restriction to $Z \times_Y Z \hookrightarrow V \times_Y V$ of the symplectic pairing.

Therefore, for any $y \in Y$, if we denote by $(-)_y$ the base change by the inclusion $\{y\} \hookrightarrow Y$, we have

$$C'_y = R\overline{pr}_{2,y,!}(\mathcal{L}_\psi(\overline{\langle,\rangle}_y) \otimes \overline{pr}_{1,y}^*C_y)[2d](d)$$

([SGA4](XVII, (5.2.6))). In particular, we have

$$C'_y \in ob\ {}^pD_c^{[-\infty, 2\dim Z_y - 2d]}(Z_y, \overline{\mathbf{Q}}_\ell)$$

([B-B-D](4.2.4)).

But it follows from [SGA 4 $\frac{1}{2}$] [Thm. finitude] (1.9) that C'_y is perverse on Z_y for any y in an open dense subset of the image of $Supp(C')$ by the canonical projection $Z' \to Y$. So the hypothesis $\dim Z_y < d$ implies $Supp(C') = \phi$, $C' = 0$, $\mathcal{F}B = 0$ and $B = 0$ and the theorem is proved. ∎

In general, ν^* is not an equivalence of categories. Let

$$C = C_\psi$$

be the full subcategory of $Perv(Z,\overline{\mathbf{Q}}_\ell)$ With

$$ob\, C = \{C \in ob\, Perv(Z,\overline{\mathbf{Q}}_\ell) \mid j^*\mathcal{F}i_*C = 0\}$$

and let

$$\mu_* = \mu_{\psi,*} : C \to Perv(V,\overline{\mathbf{Q}}_\ell)$$

be the composition of the inclusion of C into $Perv(Z,\overline{\mathbf{Q}}_\ell)$ with i_*. Then we have:

LEMMA 1.5.5. (i) *The functor is fully faithful and identifies* C *with a thick (i.e., stable by extensions and subquotients) abelian subcategory of* $Perv(V,\overline{\mathbf{Q}}_\ell)$.

(ii) *The functor* μ_* *admits a left adjoint functor*

$$\mu^* = \mu_\psi^* : Perv(V,\overline{\mathbf{Q}}_\ell) \to C$$

and a right adjoint functor

$$\mu^! = \mu_\psi^! : Perv(V,\overline{\mathbf{Q}}_\ell) \to C;$$

moreover μ^* *is right exact,* $\mu^!$ *is left exact and there exists a canonical morphism of functors*

$$\mu^! \to \mu^*$$

such that the following diagrams commute

$$\begin{array}{ccc} id & \to & id \\ \downarrow & & \uparrow \\ \mu^!\mu_* & \to & \mu^*\mu_* \end{array} \quad \textit{and} \quad \begin{array}{ccc} id & \to & id \\ \uparrow & & \downarrow \\ \mu_*\mu^! & \to & \mu_*\mu^* \end{array}$$

(the two top horizontal arrows are identities, the two bottom horizontal arrows are induced by $\mu^! \to \mu^*$ *and the four vertical arrows are adjunction maps).*

(iii) *The functors* $\mathcal{F}_V$ *and* D_V *on* $Perv(V,\overline{\mathbf{Q}}_\ell)$ *induce functors*

$$\mathcal{F} = \mathcal{F}_\psi : C \to C$$

and

$$\mathcal{D} = \mathcal{D}_{\psi} : \mathcal{C}_{\psi} \to \mathcal{C}_{\psi^{-1}},$$

such that

$$\mathcal{F}^2 \simeq id$$
$$\mathcal{D}_{\psi^{-1}}\mathcal{D}_{\psi} \simeq id$$

and

$$\mathcal{D}_{\psi}\mathcal{F}_{\psi} \simeq \mathcal{F}_{\psi^{-1}}\mathcal{D}_{\psi};$$

moreover, $\mathcal{F}$ *commutes with* $\mu^*, \mu_*, \mu^!$ *and the canonical map* $\mu^! \to \mu^*$ *and* $\mathcal{D}_{\psi}$ *exchanges* μ^*_{ψ} *and* $\mu^!_{\psi^{-1}}, \mu_{\psi,*}$ *and* $\mu_{\psi^{-1},*}$, $\mu^!_{\psi}$ *and* $\mu^*_{\psi^{-1}}$ *and the canonical maps* $\mu^!_{\psi} \to \mu^*_{\psi^{-1}}$ *and* $\mu^!_{\psi^{-1}} \to \mu^*_{\psi^{-1}}$.

(iv) *The adjunction maps*

$$\mu^*\mu_* \to id$$
$$id \to \mu^!\mu_*$$

are isomorphisms.

Proof. The proof is straightforward. We will give formulas for $\mu^*, \mu^!$ and $\mu^! \to \mu^*$.

For any perverse sheaf B on V, we have functorial maps in $Perv(V, \overline{\mathbf{Q}}_\ell)$

$${}^p j_! j^* B \oplus \mathcal{F}{}^p j_! j^* \mathcal{F} B \to B$$

and

$$B \to {}^p j_* j^* B \oplus \mathcal{F}{}^p j_* j^* \mathcal{F} B$$

given by the following matrices

$$M^* = (({}^p j_! j^* \to id)(B), (\mathcal{F}{}^p j_! j^* \mathcal{F} \to id)(B))$$

and

$$M^! = \begin{pmatrix} (id \to {}^p j_* j^*)(B) \\ (id \to \mathcal{F}{}^p j_* j^* \mathcal{F})(B) \end{pmatrix}.$$

It is easy to check that

$$j^*Coker(M^*) = j^*Ker(M^!) = 0$$

and

$$j^*\mathcal{F}\ Coker(M^*) = j^*\mathcal{F}\ Coker(M^!) = 0,$$

so that we can define μ^*B and $\mu^!B$ by the formulas

$$\mu_*\mu^*B = Coker(M^*)$$

and

$$\mu_*\mu^!B = Ker(M^!).$$

Moreover, the map

$$\mu_*\mu^!B = Ker(M^!) \hookrightarrow B \twoheadrightarrow Coker(M^*) = \mu_*\mu^*B$$

comes from a map

$$\mu^!B \to \mu^*B$$

which is the one we want. ∎

COROLLARY 1.5.6. (i) *For any perverse sheaf B on V, the sequences given by adjunction maps*

$$\nu_!\nu^*B \to B \to \mu_*\mu^*B \to 0$$

$$0 \to \mu_*\mu^!B \to B \to \nu_*\nu^*B$$

are exact.

(ii) *The functors*

$$\nu^*\mu_* : \mathcal{C} \to \mathcal{A}$$

and

$$\mu^*\nu_!, \mu^!\nu_* : \mathcal{A} \to \mathcal{C}$$

are identically zero.

(iii) *The functor ν^* identifies the category $\mathcal{A}$ with the quotient of the abelian category $Perv(V, \overline{\mathbf{Q}}_\ell)$ by its thick subcategory $\mathcal{C}$ (or more precisely $\mu_*(\mathcal{C})$).*

Proof. Part (i) follows immediately from our formulas for $\mu_*\mu^*B$ and $\mu_*\mu^!B$ and the fact that $M^*N_!(\nu^*)$ and $N_*(\nu^*)M^!$ are identically zero.

The fact that $\nu^*\mu_* = 0$ is obvious and by duality $\mu^!\nu_* = 0$ follows from $\mu^*\nu_! = 0$. Now, if $(A \frac{\alpha}{\beta} A') \in ob\,\mathcal{A}$ and $\nu_!(A \frac{\alpha}{\beta} A') = B$, we have

$$j^*B = A$$
$$j^*\mathcal{F}B = A'$$

($\nu^*\nu_! = id$, cf. (1.5.2)), so that

$$M^* : {}^pj_!j^*B \oplus \mathcal{F}^pj_!j^*B \to B$$

is nothing else than the quotient map

$${}^pj_!A \oplus \mathcal{F}^pj_!A' \twoheadrightarrow Coker(N_!) = \nu_!(A \frac{\alpha}{\beta} A')$$

and

$$\mu_*\mu^*\nu_!(A \frac{\alpha}{\beta} A') = Coker(M^*) = 0.$$

The part (iii) follows immediately from the parts (i) and (ii). ∎

In some particular cases, C can be described as the category of perverse sheaves on a scheme of finite type over k. Here is an example:

PROPOSITION 1.5.7. *Let us assume that $Z \to Y$ is a coisotropic, rank r, vector subbundle of the symplectic vector bundle $V \xrightarrow{\pi} Y$, i.e., $Z^\perp \subset Z$ where $Z^\perp \overset{i^\perp}{\hookrightarrow} V$ is the orthogonal subbundle to $Z \overset{i}{\hookrightarrow} V$. Let*

$$\pi_1 : V_1 = Z/Z^\perp \to Y$$

be the corresponding quotient bundle and let

$$f : Z \twoheadrightarrow V_1$$

be the quotient map. Then, for any perverse sheaf B_1 on V_1, the perverse sheaf

$$\tilde{f}B = f^*B[2d-r](d) = f^!B[r-2d](r-d)$$

on Z is in fact an object of C and the functor

$$\tilde{f} : Perv(V_1, \overline{\mathbf{Q}}_\ell) \to C$$

is an equivalence of abelian categories.

Proof. Let $g : V \to V/Z^\perp$ be the quotient map and $\bar{i} : V_1 = Z/Z^\perp \hookrightarrow V/Z^\perp$ the map induced by i. Then we have a cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{i} & V \\ {\scriptstyle f}\downarrow & \square & \downarrow{\scriptstyle g} \\ V_1 & \xrightarrow{\bar{i}} & V/Z^\perp \end{array}$$

But, the symplectic structure on V induces a symplectic structure on V_1 and the above commutative diagram is autodual. Therefore,

$$\mathcal{F}i_*(ob\, Perv(Z,\overline{\mathbf{Q}}_\ell)) \subset \tilde{g}(ob\, Perv(V/Z^\perp))$$

where

$$\tilde{g}(-) = g^*(-)[2d-r](d) = g^!(-)[r-2d](r-d)$$

and we have a canonical isomorphism of functors from $Perv(V_1,\overline{Q}_\ell)$ to $Perv(V,\overline{Q}_\ell)$

$$\mathcal{F}i_*\tilde{f} \simeq i_*\tilde{f}\mathcal{F}_1$$

([La] (1.2.2.4) and (1.2.2.1), $\tilde{g}\bar{i}_* \simeq i_*\tilde{f}$ by base change).

As the functor

$$\tilde{f} : Perv(V_1,\overline{\mathbf{Q}}_\ell) \to Perv(Z,\overline{\mathbf{Q}}_\ell)$$

is fully-faithful ([B-B-D] (4.2.5)), the proposition follows easily from these facts. ∎

2. GLUING FOR G/U

(2.0) The notations and the assumptions of (1.0), (1.1) and (1.2) are enforced. We will assume moreover that $p > 2$.

Let G be a connected, simply connected, semi-simple algebraic group over k. We will assume that G is split over k and we will fix a maximal torus T, split over k, and a Borel subgroup B of G containing T.

Let $(\mathcal{X}, R, \mathcal{X}^V, R^V)$ be the root datum associated to (G,T), W the corresponding Weyl group, R^+ the system of positive roots in R associated to B, Π the set of the simple positive roots determined by R^+ and S the corresponding set of simple reflexions in W. We will denote by $\alpha \mapsto \alpha^V$ the canonical bijection of R onto R^V and by Π^V the image of Π by this canonical bijection. If we denote by α_s the simple positive root corresponding to $s \in S$, we have

$$\Pi = \{\alpha_s \mid s \in S\}$$

and

$$\Pi^V = \{\alpha_s^V \mid s \in S\}.$$

The group W is finite and (W, S) is a Coxeter system; in particular, W has a length function $\ell : W \to \mathbf{N}$; for each $w \in W, \ell(w)$ is the number of elements in

$$R(w) = \{\alpha \in R^+ \mid w\alpha \in -R^+\}.$$

We will denote by w_0 the longest element in W.

As G is simply connected, $\mathcal{X}^V = X_*(T)$ coincides with the root lattice $\mathbf{Z}R^V$ and Π^V is a basis of $\mathcal{X}^V$. Let $(\omega_s)_{s\in S}$ be the basis of $\mathcal{X}$ dual to $\Pi^V = \{\alpha_s^V \mid s \in S\}$ (the fundamental weights); for each $\lambda \in \mathcal{X}$, we have

$$\lambda = \sum_{s\in S} \langle \alpha_s^V, \lambda\rangle \omega_s.$$

For each $\alpha \in R$, let X_α be the root subgroup of G corresponding to α. If U is the unipotent radical of B, we have

$$\prod_{\alpha\in R^+} X_\alpha \xrightarrow{\sim} U$$

where the map is induced by the group law of G. More generally, if, for each $w \in W$, we set

$$U_w = U \cap \dot{w}^{-1}\dot{w}_0^{-1} U \dot{w}_0 \dot{w}$$

($\dot{w}_0, \dot{w}$ are arbitrary representative of w_0, w respectively in $N_G(T)$), we have

$$\prod_{\alpha\in R(w)} X_\alpha \xrightarrow{\sim} U_w$$

and

$$U_w \times U_{w_0 w} \xrightarrow{\sim} U.$$

The X_α 's and the U_w 's are normalized by T. If $w = s \in S$, $R(w) = \{\alpha_s\}$ and $U_w = X_{\alpha_s}$. For a general $w \in W$, dim $U_w = \ell(w)$ $(U_1 = \{1\}, U_{w_0} = U)$.

For each $s \in S$, we have a parabolic subgroup P_s of G containing B of semi-simple rank one canonically attached to s,

$$P_s = \langle X_{-\alpha_s}, B\rangle;$$

P_s has a Levi decomposition

$$P_s = L_s \cdot U_{w_0 s}$$

where

$$L_s = \langle X_{-\alpha_s}, X_{\alpha_s} \rangle \cdot Ker(\alpha)$$

and

$$\langle X_{-\alpha_s}, X_{\alpha_s} \rangle \cap Ker(\alpha) = \{\pm 1\}.$$

We set

$$M_s = \langle X_{-\alpha_s}, X_{\alpha_s} \rangle;$$

M_s, which is nothing else than the commutator subgroup (L_s, L_s) of L_s, is a connected simple algebraic group over k of rank 1 and, from our hypotheses on G, M_s is simply connected and split over k;

$$T_s = T \cap M_s$$

is a split maximal torus of M_s and

$$\alpha_s^V : \mathbf{G}_{m,k} \xrightarrow{\sim} T_s \subset T$$

is an isomorphism with inverse induced by ω_s. Such a group is non canonically isomorphic to $SL_{2,k}$ by an isomorphism

$$\varphi_s : M_s \xrightarrow{\sim} SL_{2,k}$$

satisfying the conditions

$$\varphi_s^{-1}\left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}\right) = \alpha_s^V(z)$$

and

$$\varphi_s^{-1}\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\right) = x_{-s}(u), \quad \varphi_s^{-1}\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) = x_s(u)$$

where

$$x_{-s} : \mathbf{G}_{a,k} \xrightarrow{\sim} X_{-\alpha_s}, \; x_s : \mathbf{G}_{a,k} \xrightarrow{\sim} X_{\alpha_s} = U_s$$

are isomorphisms of 1-dimensional additive algebraic groups over k; x_{-s} and consequently φ_s is uniquely determined by x_s.

For each $s \in S$, *we fix an isomorphism*

$$x_s : \mathbf{G}_{a,k} \xrightarrow{\sim} U_s.$$

We set

$$n_s = \varphi_s^{-1}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \in N_{M_s}(T_s),$$

where φ_s is determined by x_s; in $N_G(T)$ which contains obviously $N_{M_s}(T_s)$, n_s is a representative of s.

We set

$$Q_s = M_s U_{w_0 s}$$

for each $s \in S$; Q_s is nothing else than the commutator subgroup (P_s, P_s) of P_s.

For more details, the reader can look at Springer's book [Sp].

(2.1) Let

$$X = G/U$$

and, for each $s \in S$, let

$$\tau_s : X \to Y_s = G/Q_s$$

be the canonical projection $(U \subset Q_s)$.

We can identify X with the complementary open subset of the zero section in a G-equivariant rank 2 vector bundle over Y_s

$$\pi_s : V_s \to Y_s.$$

This vector bundle is defined in the following way. M_s acts by right translation on $G/U_{w_0 s}$ (M_s normalizes $U_{w_0 s}$) and $G/U_{w_0 s}$ is an M_s-torsor over $G/Q_s = Y_s$. But, now, the isomorphism $\varphi_s : M_s \to SL_{2,k}$ defines a left action of M_s on the vector space $k^2 = \left\{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right\}$ and we can take

$$V_s = G/U_{w_0 s} \overset{M_s}{\times} k^2,$$

with the natural G-action by left translation. The subgroup U_s of M_s is the stabilizer of $\begin{pmatrix}1\\0\end{pmatrix}$ in k^2 and the map

$$G \to (G/U_{w_0 s}) \times k^2, \quad g \mapsto \left(gU_{w_0 s}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

induces a G-equivariant open embedding

$$j_s : X \hookrightarrow V_s$$

which identifies X with the complementary open subset of the zero section $Z_s \overset{i_s}{\hookrightarrow} V_s$ in V_s.

As the action of M_s on k^2 fixes the volume form $dv_1 \wedge dv_2$, V_s has a canonical G-invariant symplectic structure

$$\langle , \rangle_s : V_s \times_{Y_s} V_s \to \mathbf{G}_{a,k}$$

and the restriction to $X \times_{Y_s} X$ of $\langle , \rangle_s$ is given by

$$\langle g_1 U, g_2 U\rangle_s = c$$

where

$$g_1^{-1} g_2 = mu \in M_s U_{w_0 s} = Q_s$$

and

$$\varphi_s(m) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(2.2) For each $w \in W$, we set

$$T_w = \prod_{\alpha \in R(w)} \alpha^V(\mathbf{G}_m) \subset T;$$

T_w is a torus and we have an obvious surjective homomorphism of tori

$$\prod_{\alpha \in R(w)} \alpha^V : \mathbf{G}_{m,k}^{R(w)} \twoheadrightarrow T_w.$$

LEMMA 2.2.1. *Let* $S_w = \{s \in S \mid s \leq w\}$ *($\leq$ is the Bruhat order on W), then the sub-torus* $T_w \subset T$ *is equal to*

$$\prod_{s \in S_w} T_s \subset T.$$

In other words, we have an isomorphism

$$\prod_{s \in S_w} \alpha_s^{\vee} : \mathbf{G}_{m,k}^{S_w} \xrightarrow{\sim} T_w \subset T$$

with inverse induced by $\prod_{s \in S_w} \omega_s$.

Proof. By induction on $\ell(w)$. For each $w \in W$ and each $s \in S$ such that $\ell(ws) = \ell(w) + 1$, we have

$$R(ws) = s(R(w)) \cup \{\alpha_s\}$$

and consequently

$$T_{ws} = (T_w)^{n_s} T_s.$$

Now, if

$$T_w = \prod_{s' \in S_w} T_{s'},$$

we have

$$T_{ws} = \Big(\prod_{s' \in S_w} (T_{s'})^{n_s}\Big) \cdot T_s.$$

But

$$(T_{s'})^{n_s} \cdot T_s = T_{s'} \cdot T_s$$

as

$$s\,\alpha_{s'}^{\vee} = \alpha_{s'}^{\vee} - \langle \alpha_{s'}^{\vee}, \alpha_s \rangle \alpha_s^{\vee}$$

for any $s' \in S'$. So the lemma is proved. ∎

For each $w \in W$ with shortest expression

$$w = s_1 \cdots s_\ell$$

as a product of simple reflexions ($\ell = \ell(w)$), we set

$$n_w = n_{s_1} \cdots n_{s_\ell};$$

it is known that n_w depends only on w and not on the choice of the shortest expression ([Sp] (11.2.9)).

LEMMA 2.2.2. (i) *For each $w \in W$ and each $s \in S$ such that $\ell(ws) = \ell(w) + 1$ we have*

$$(U\, n_w T_w U)(U\, n_s T_s U) = U\, n_{ws} T_{ws} U.$$

(ii) *For each shortest expression*

$$w = s_1 \cdots s_\ell$$

of $w \in W$, we have

$$U\, n_w T_w U = (U\, n_{s_1} T_{s_1} U) \cdots (U\, n_{s_\ell} T_{s_\ell} U).$$

Proof. The part (ii) follows trivially from the part (i). The part (i) follows from

$$U\, n_w T_w U = U\, n_w T_w U_w$$

($n_w T_w$ normalizes $U_{w_0 w}$) and from the induction formulas

$$n_{ws} = n_w n_s$$
$$T_{ws} = (T_w)^{n_s} T_s$$

and

$$U_{ws} = (U_w)^{n_s} U_s$$

$(R(w\, s) = s(R(w)) \cup \{\alpha_s\})$. ■

DEFINITION 2.2.3. *For each $w \in W$, let*

$$X(w) \subset X \times_k X$$

be the subvariety defined by

$$X(w) = \{(gU, g'U) \in X \times_k X \mid g^{-1}g' \in U\, n_w T_w U\}$$

and let

$$pr_w : X(w) \to T_w$$

be the projection sending $(gU, g'U)$ *to the unique* $t_w \in T_w$ *such that*

$$g^{-1}g' \in U\, n_w t_w U.$$

It is clear that $X(w)$ is a connected, smooth, locally closed subvariety of $X \times_k X$ of dimension

$$\dim\ X + \ell(w) + \# S_w$$

and that pr_w is smooth, surjective, with connected geometric fibers.

For $w = 1, X(w) = X$ diagonally embedded in $X \times_k X$.

For $s \in S$, we have

$$X(s) = \{(x, x') \in X \times_{Y_s} X \mid \langle x, x'\rangle_s \neq 0\}$$

and

$$pr_s(x, x') = -\alpha_s^\vee(\langle x, x'\rangle_s)$$

where

$$\langle , \rangle_s : X \times_{Y_s} X \to \mathbf{G}_{a,k}$$

is the restriction of the symplectic pairing on $V_s \to Y_s$ (we have

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with

$$c = -z$$

in $SL_{2,k}$; cf. (2.1)).

LEMMA 2.2.4. *For each* $w \in W$ *with shortest expression*

$$w = s_1 \cdots s_\ell$$

we have a cartesian square

$$\begin{array}{ccc} X(s_1, \cdots, s_\ell) & \xrightarrow{pr_{s_1,\cdots,s_\ell}} & \mathbf{G}_{m,k}^{R(w)} \\ {\scriptstyle pr_{0\ell}}\downarrow & & \downarrow \prod_{\alpha\in R(w)} \alpha^\vee \\ X(w) & \xrightarrow{pr_w} & T_w \end{array}$$

where

$$X(s_1,\cdots,s_\ell) = X(s_1)\times_X X(s_2)\times_X\cdots\times_X X(s_\ell)$$
$$= \{(g_0U,\cdots,g_\ell U)\in X^{\ell+1}\mid \forall i=1,\cdots,\ell,$$
$$g_{i-1}^{-1}g_i\in U\, n_iT_iU\},$$

$$pr_{s_1,\cdots,s_\ell}(g_0U,\cdots,g_\ell U) = (pr_{s_i}(g_{i-1}U,g_iU))_{i=1,\dots,\ell}$$

(we have

$$R(w) = \{s_\ell\cdots s_2\alpha_1,\cdots,s_\ell\alpha_{\ell-1},\alpha_\ell\}$$

where $\alpha_1,\cdots,\alpha_\ell$ *are the simple positive roots corresponding to* $s_1,\cdots,s_\ell$ *respectively* ([Sp] (10.2.2)) *and we have identified for each* $i=1,\cdots,\ell$,

$$T_i = T_{s_i}$$

with $\mathbf{G}_{m,k}$ *via* $\alpha_s^\vee$ *and* ω_s *) and*

$$pr_{0\ell}(g_0U,\cdots,g_\ell U) = (g_0U,g_\ell U).$$

Proof. It follows from (2.2.2) that the map $pr_{o\ell}$ is well defined and surjective. Moreover, we have

$$U\, n_1\alpha_1^\vee(z_1)U\cdots U\, n_\ell\alpha_\ell^\vee(z_\ell)U = U\, n_wt_wU$$

for some $z_1,\cdots,z_\ell\in\mathbf{G}_{m,k}$ and $t_w\in T_w$ if and only if

$$(s_\ell\cdots s_2\alpha_1)^\vee(z_1)\cdots(s_\ell\alpha_{\ell-1})^\vee(z_{\ell-1})\alpha_\ell^\vee(z_\ell) = t_w$$

(we have set $n_i = n_{s_i}$ $(i=1,\dots,\ell)$). So the lemma is proved. ■

DEFINITION 2.2.5. *For each* $w\in W$ *, let*

$$\sigma_w:\mathbf{G}_{m,k}^{R(w)}\to\mathbf{G}_{a,k}$$

be the morphism defined by

$$\sigma_w((z_\alpha)_{\alpha\in R(w)}) = -\sum_{\alpha\in R(w)} z_\alpha.$$

Then, we set

$$K(w) = K_{\psi}(w) = pr_w^* R(\prod_{\alpha\in R(w)} \alpha^V)_! \mathcal{L}_{\psi}(\sigma_w)[2\ell(w)](\ell(w))$$

in $D_c^b(X(w), \overline{\mathbf{Q}}_\ell)$.

For $w = 1$, $K(w) = \overline{\mathbf{Q}}_{\ell,X}$. For any $s \in S$, we have

$$K(s) = \mathcal{L}_{\psi}(\langle , \rangle_s)[2](1)$$

where $\langle , \rangle_s : X(s) \subset X \times_{Y_s} X \to \mathbf{G}_{a,k}$ is the restriction of the symplectic pairing on V_s.

LEMMA 2.2.6. *Let*

$$\Gamma \subset T_w \times_k \mathbf{G}_{m,k}^{R(w)}$$

be the graph of $\prod_{\alpha\in R(w)} \alpha^V$. *Then* Γ, *which is obviously a connected, smooth, closed subvariety of* $T_w \times_k \mathbf{G}_{m,k}^{R(w)}$ *(isomorphic to* $\mathbf{G}_{m,k}^{R(w)}$*), is in fact also closed in* $T_w \times_k \mathbf{A}_k^{R(w)}$ *(*$\mathbf{G}_{m,k}^{R(w)}$ *is naturally embedded in* $\mathbf{A}_k^{R(w)}$*).*

Proof. We have $\langle \alpha^V, \omega_s\rangle \geq 0$, $\forall\, \alpha \in R^+$ and $s \in S$; moreover, for each $\alpha \in R(w)$, there exists $s \in S_w$ such that $\langle \alpha^V, \omega_s\rangle \neq 0$. So the lemma follows from (2.2.1). ∎

PROPOSITION 2.2.7. *For each* $w \in W$, *the canonical map*

$$R(\prod_{\alpha\in R(w)} \alpha^V)_! \mathcal{L}_{\psi}(\sigma_w) \to R(\prod_{\alpha\in R(w)} \alpha^V)_* \mathcal{L}_{\psi}(\sigma_w)$$

in $D_c^b(T_w, \overline{\mathbf{Q}}_\ell)$ *is an isomorphism. Moreover, these two isomorphic objects of* $D_c^b(T_w, \overline{\mathbf{Q}}_\ell)$ *are in fact, up to a shift of* $\ell(w)$, *irreducible perverse sheaves on* T_w.

Proof. The graph Γ of $\prod_{\alpha\in R(w)} \alpha^V$ is closed in $T_w \times_k \mathbf{A}_k^{R(w)}$ (cf. (2.2.6)). In particular, the object of $D_c^b(T_w \times_k \mathbf{A}_k^{R(w)})$,

$$\overline{\mathbf{Q}}_{\ell,\Gamma}[\ell(w)],$$

is an irreducible perverse sheaf on $T_w \times \mathbf{A}_k^{R(w)}$.

Let

$$\mathcal{F}_!, \mathcal{F}_* : D_c^b(T_w \times_k \mathbf{A}_k^{R(w)}, \overline{Q}_\ell) \to D_c^b(T_w \times_k \mathbf{A}_k^{R(w)}, \overline{\mathbf{Q}}_\ell)$$

be the Deligne-Fourier transformations relative to T_w, defined by

$$\mathcal{F}_!(-) = R\, pr_!(\mathcal{L}_\psi(-\sum_{\alpha \in R(w)} z_\alpha z'_\alpha) \otimes pr'^*(-))[\ell(w)]$$

and

$$\mathcal{F}_*(-) = R\, pr_*(\mathcal{L}_\psi(-\sum_{\alpha \in R(w)} z_\alpha z'_\alpha) \otimes pr'^*(-))[\ell(w)]$$

where

$$pr, pr' : T_w \times_k \mathbf{A}_k^{R(w)} \times_k \mathbf{A}_k^{R(w)} \to T_w \times_k \mathbf{A}_k^{R(w)}$$

are the canonical projections. It is known that the canonical map

$$\mathcal{F}_!(-) \to \mathcal{F}_*(-)$$

is an isomorphism of functors and that $\mathcal{F}_!(-)$ induces an equivalence of abelian categories between the subcategories of perverse sheaves ([La] (1.3.1.1) and (1.3.2.3)).

Now, let

$$m : T_w \times_k \mathbf{G}_{m,k}^{R(w)} \to T_w$$

be the multiplication map,

$$m(t_w, (z_\alpha)_{\alpha \in R(w)}) = t_w \prod_{\alpha \in R(w)} \alpha^V(z_\alpha).$$

Then it follows easily from the proper and smooth base change theorems ([SGA 4] (XVII, (5.2.6)) and (XVI, (1.1))) that we have a commutative diagram in $D_c^b(T_w \times_k \mathbf{G}_{m,k}^{R(w)})$

$$\begin{array}{ccc} m^* R(\prod_{\alpha \in R(w)} \alpha^V)_! \mathcal{L}_\psi(\sigma_w)[2\ell(w)] & \xrightarrow{\sim} & \mathcal{F}_!(\overline{\mathbf{Q}}_{\ell,\Gamma}[\ell(w)])_{|T_w \times_k \mathbf{G}_{m,k}^{R(w)}} \\ \downarrow & & \downarrow \\ m^* R(\prod_{\alpha \in R(w)} \alpha^V)_* \mathcal{L}_\psi(\sigma_w)[2\ell(w)] & \xrightarrow{\sim} & \mathcal{F}_*(\overline{\mathbf{Q}}_{\ell,\Gamma}[\ell(w)])_{|T_w \times_k \mathbf{G}_{m,k}^{R(w)}} \end{array}$$

where the horizontal arrows are isomorphisms and the vertical ones are canonical (in terms of functions, we have

$$\sum_{\prod_{\alpha\in R(w)}\alpha^{\vee}(y_\alpha)=t_w \prod_{\alpha\in R(w)}\alpha^{\vee}(z_\alpha)} \psi\left(-\sum_{\alpha\in R(w)} y_\alpha\right) =$$

$$= \sum_{\substack{(z'_\alpha)_{\alpha\in R(w)} \\ \prod_{\alpha\in R(w)}\alpha^{\vee}(z'_\alpha)=t_w}} \psi\left(-\sum_{\alpha\in R(w)} z_\alpha z'_\alpha\right)$$

by the change of variables $z'_\alpha = y_\alpha/z_\alpha$). As m is smooth with connected geometric fibers, an application of [B-B-D] (4.2.5) finishes the proof of the proposition. ∎

COROLLARY 2.2.8. *For each $w \in W$, the canonical map in $D^b_c(X(w),\overline{\mathbf{Q}}_\ell)$*

$$K(w) \to pr^*_w R(\prod_{\alpha\in R(w)}\alpha^{\vee})_*\mathcal{L}_\psi(\sigma_w)[2\ell(w)](\ell(w))$$

is an isomorphism and

$$K(w)[\dim X]$$

is an irreducible perverse sheaf on $X(w)$.

Proof. Apply [B-B-D] (4.2.5). ∎

COROLLARY 2.2.9. *For each $w \in W$,*

$$D\,K_\psi(w) = K_{\psi^{-1}}(w)[2\ \dim\ X](\dim\ X)$$

where D is the duality functor on $D^b_c(X(w),\overline{\mathbf{Q}}_\ell)$.

PROPOSITION 2.2.10. *For each $w \in W$ with shortest expression*

$$w = s_1\cdots s_\ell$$

the canonical map in $D^b_c(X(w),\overline{\mathbf{Q}}_\ell)$

$$R\,pr_{0\ell!}(pr^*_{01}K(s_1)\otimes\cdots\otimes pr^*_{\ell-1,\ell}K(s_\ell))$$

$$\to R\,pr_{0\ell*}(pr^*_{01}K(s_1)\otimes\cdots\otimes pr^*_{\ell-1,\ell}K(s_\ell))$$

where

$$pr_{i-1,1} : X(s_1,\cdots,s_\ell) \to X(s_i)$$

maps $(g_0U,\cdots,g_\ell U)$ to $(g_{i-1}U,g_iU)$ $(i = 1,\cdots,\ell)$, is an isomorphism. Moreover, these two isomorphic objects of $D^b_c(X(w),\overline{\mathbf{Q}}_\ell)$ are in fact isomorphic to $K(w)$.

Proof. This is an easy consequence of (2.2.4), the proper and the smooth base change theorems ([SGA 4] (XVII, (5.2.6)) and (XVI, (1.1)) and (2.2.8). ∎

(2.3) For each $w \in W$, let $\overline{X(w)}$ be the closure of $X(w)$ inside $X \times_k X$ and let

$$\nu_w : X(w) \hookrightarrow \overline{X(w)}$$

be the inclusion. We set

(2.3.1) $$\overline{K(w)} = \overline{K_\psi(w)} = \nu_{w!*}K(w).$$

($K(w)$ is a perverse sheaf on $X(w)$ up to a shift of dim X). It follows from (2.2.8) and (2.2.9):

PROPOSITION 2.3.2. *For each $w \in W$, $\overline{K(w)}[\dim X]$ is an irreducible perverse sheaf on $X(w)$ and*

$$D\overline{K_\psi(w)} = \overline{K_{\psi^{-1}}(w)}[2 \dim X](\dim X)$$

where D is the duality functor on $D_c^b(\overline{X(w)}, \overline{\mathbf{Q}}_\ell)$. ∎

For $w = 1$, $\overline{X(w)} = X$ diagonally embedded in $X \times_k X$ and $\overline{K(w)} = \overline{\mathbf{Q}}_{\ell,X}$. For any $s \in S$, we have

$$\overline{X(s)} = X \times_{Y_s} X$$

and

$$\overline{K(s)} = \mathcal{L}_\psi(\langle,\rangle_s)[2](1),$$

where

$$\langle,\rangle_s : X \times_{Y_s} X \to \mathbf{G}_{a,k}$$

is the restriction of the symplectic pairing on V_s (cf. (2.1)). For a general $w \in W$ with shortest expression

$$w = s_1 \cdots s_\ell,$$

we have a partial compactification

$$\nu_{s_1,\cdots,s_\ell} : X(s_1,\cdots,s_\ell) \hookrightarrow \overline{X(s_1,\cdots,s_\ell)}$$

where

$$\overline{X(s_1,\cdots,s_\ell)} = \overline{X(s_1)} \times_X \overline{X(s_2)} \times_X \cdots \times_X \overline{X(s_\ell)}$$

and a natural extension

$$\overline{pr}_{0\ell} : \overline{X(s_1,\cdots,s_\ell)} \to \overline{X(w)}.$$

It is not difficult to check that:

LEMMA 2.3.3. (i) $\overline{X(s_1,\cdots,s_\ell)}$ *is a connected smooth, quasi-projective variety of dimension* dim $X + 2\ell$ *over* k *and*

$$\overline{X(s_1,\cdots,s_\ell)} - X(s_1,\cdots,s_\ell)$$

is a divisor with normal crossings which is in fact the union of ℓ *irreducible smooth divisors*

$$\begin{aligned}&\overline{X(s_1)} \times_X \cdots \times_X \overline{X(s_{i-1})} \times_X (\overline{X(s_i)} - \\ &- X(s_i)) \times_X \overline{X(s_{i+1})} \times_X \cdots \times_X \overline{X(s_\ell)} \qquad (i = 1,\cdots,\ell).\end{aligned}$$

(ii) *The commutative square*

$$\begin{array}{ccc} X(s_1,\cdots,s_\ell) & \overset{\nu_{s_1,\cdots,s_\ell}}{\to} & \overline{X(s_1,\cdots,s_\ell)} \\ {\scriptstyle pr_{0\ell}}\downarrow & & \downarrow{\scriptstyle \overline{pr}_{0\ell}} \\ X(w) & \overset{\nu_w}{\to} & \overline{X(w)} \end{array}$$

is in fact cartesian. ∎

THEOREM 2.3.4. *For each* $w \in W$ *with shortest expression*

$$w = s_1 \cdots s_\ell,$$

the canonical map

$$\begin{aligned}&R\,\overline{pr}_{0\ell!}(\overline{pr}^*_{01}\overline{K(s_1)} \otimes \cdots \otimes \overline{pr}^*_{\ell-1,\ell}\overline{K(s_\ell)}) \\ &\qquad \to R\,\overline{pr}_{0\ell*}(\overline{pr}^*_{01}\overline{K(s_1)} \otimes \cdots \otimes \overline{pr}^*_{\ell-1,\ell}\overline{K(s_\ell)})\end{aligned}$$

is an isomorphism in $D^b_c(\overline{X(w)}, \mathbf{Q}_\ell)$, *where*

$$\overline{pr}_{i-1,i} : \overline{X(s_1,\cdots,s_\ell)} \to \overline{X(s_i)}$$

is defined by

$$\overline{pr}_{i-1,i}(g_0U,\cdots,g_\ell U) = (g_{i-1}U, g_iU)$$

for $i = 1,\ldots,\ell$; *moreover these two canonically isomorphic objects of* $D^b_c(\overline{X(w)}$, $\overline{\mathbf{Q}}_\ell)$ *are in fact isomorphic to* $\overline{K(w)}$.

The proof of Theorem 2.3.4 will be given in (2.4) and (2.5).

(2.4) Let

$$\mathcal{B} = G/B$$

be the variety of Borel subgroups of G; it is a connected, smooth and projective variety over k. G acts by left translation on $\mathcal{B}$ and consequently on $\mathcal{B} \times_k \mathcal{B}$ and the orbits of this action on $\mathcal{B} \times_k \mathcal{B}$ are parametrized by W: for each $w \in W$,

$$O(w) = \{(gB, g'B) \in \mathcal{B} \times_k \mathcal{B} | g^{-1}g' \in Bn_w B\}$$

is an orbit. For each $w \in W$, $O(w)$ is a connected, smooth, locally closed subvariety of $\mathcal{B} \times_k \mathcal{B}$ of dimension

$$\dim O(w) = \dim \mathcal{B} + \ell(w)$$

and its closure

$$\overline{O(w)} \subset \mathcal{B} \times_k \mathcal{B}$$

is the disjoint union of the orbits $O(w')$ for $w' \leq w$ in W.

In general $\overline{O(w)}$ is singular but for each shortest expression

$$w = s_1 \cdots s_\ell$$

we have a canonical resolution of singularities

$$\overline{pr}_{0\ell} : \overline{O(s_1, \cdots, s_\ell)} \to \overline{O(w)}$$

where

$$\overline{O(s_1, \cdots, s_\ell)} = \overline{O(s_1)} \times_{\mathcal{B}} \overline{O(s_2)} \times_{\mathcal{B}} \times \cdots \times_{\mathcal{B}} \overline{O(s_\ell)}$$

and

$$pr_{0\ell}(g_0 B, \cdots, g_\ell B) = (g_0 B, g_\ell B)$$

(cf. [Dem] 3).

Now, we have an obvious commutative diagram

$$\begin{array}{ccc} \overline{X(s_1, \cdots, s_\ell)} & \to & \overline{O(s_1, \cdots, s_\ell)} \\ {\scriptstyle pr_{0\ell}} \downarrow & & \downarrow {\scriptstyle \overline{pr}_{0\ell}} \\ \overline{X(w)} & \to & \overline{O(w)} \end{array}$$

where the horizontal arrows are given by

$$(g_0U,\cdots,g_\ell U)\mapsto(g_0B,\cdots,g_\ell B)$$

and

$$(gU,g'U)\mapsto(gB,g'B).$$

In particular, we have a factorization of $\overline{pr}_{0\ell}:\overline{X(s_1,\cdots,s_\ell)}\to\overline{X(w)}$ throughout the fiber product

$$\overline{X(s_1,\cdots,s_\ell)}\xrightarrow{\overline{pr}'_{0\ell}}\overline{X(w)}\times_{\overline{O(w)}}\overline{O(s_1,\cdots,s_\ell)}\xrightarrow{\overline{pr}''_{0\ell}}\overline{X(w)}.$$

On $\overline{X(s_1,\cdots,s_\ell)}$ we have an action of $\mathbf{G}_{m,k}^{R(w)}=\mathbf{G}_{m,k}^{\ell}$ ($R(w)=\{s_\ell\cdots s_2\alpha_1,\cdots,s_\ell\alpha_{\ell-1},\alpha_\ell\}$ given by

$$((g_0U,\cdots,g_\ell U),(z_1,\cdots,z_\ell))\mapsto(g_0t_0U,\cdots,g_\ell t_\ell U)$$

where

$$t_i=(s_i\cdots s_2\alpha_1)^V(z_1)\cdots(s_i\alpha_{i-1})^V(z_{i-1})\alpha_i^V(z_i)$$

for $i=0,\ldots,\ell\,(t_0=1)$: by definition of the t_i 's, we have

$$s_i(t_{i-1})^{-1}t_i\in T_i\quad(i=1,\cdots,\ell),$$

but this is equivalent to

$$t_{i-1}^{-1}t_i\in T_i\quad(i=1,\ldots,\ell)$$

as

$$s(t)^{-1}t\in T_s$$

for any $s\in S$ and any $t\in T$. Similarly, we have an action of T_w on $X(w)$ given by

$$((gU,g'U),t_w)\mapsto(gU,g't_wU).$$

The morphism

$$\overline{pr}_{0\ell}:\overline{X(s_1,\cdots,s_\ell)}\to\overline{X(w)}$$

is equivariant for these actions via the surjective homomorphism

$$\prod_{\alpha\in R(w)}\alpha^V:\mathbf{G}_{m,k}^{R(w)}\to T_w\,((z_1,\cdots,z_\ell)\mapsto t_\ell).$$

LEMMA 2.4.1. (i) *The morphism*

$$\overline{pr}'_{0\ell} : \overline{X(s_1, \cdots, s_\ell)} \to \overline{X(w)} \times_{\overline{O(w)}} \overline{O(s_1, \cdots, s_\ell)}$$

is a principal homogeneous space under the diagonalizable group

$$\mathrm{Ker}\Big(\prod_{\alpha \in R(w)} \alpha^V : \mathbf{G}_{m,k}^{R(w)} \to T_w\Big).$$

(ii) *The morphism*

$$\overline{pr}''_{0\ell} : \overline{X(w)} \times_{\overline{O(w)}} \overline{O(s_1, \cdots, s_\ell)} \to \overline{X(w)}$$

is projective.

Proof. We have

$$(g_0 U, \cdots, g_\ell U) \in \overline{X(s_1, \cdots, s_\ell)}$$

if and only if

$$g_{i-1}^{-1} g_i \in U n_i T_i U \quad \text{or} \quad U\, T_i\, U$$

for $i = 1, \cdots, \ell$. Two points $(g_0 U, \cdots, g_\ell U)$ and $(g'_0 U, \cdots, g'_\ell U)$ of $\overline{X(s_1, \cdots, s_\ell)}$ have the same image in

$$\overline{X(w)} \times_{\overline{O(w)}} \overline{O(s_1, \cdots, s_\ell)}$$

if and only if

$$g'_i \in g_i t_i U$$

for some $t_i \in T$ $(i = 0, \ldots, \ell)$ with $t_0 = t_\ell = 1$ and these t_i's are uniquely determined; moreover, we have necessarily, for each $i = 1, \ldots, \ell$,

$$-\text{either} \quad g_{i-1}^{-1} g_i, g'^{-1}_{i-1} g'_i \in U n_i T_i U \quad \text{and}$$
$$s_i(t_{i-1})^{-1} t_i \in T_i$$
$$-\text{either} \quad g_{i-1}^{-1} g_i, g'^{-1}_{i-1} g'_i \in U\, T_i U \quad \text{and}$$
$$t_{i-1}^{-1} t_i \in T_i.$$

As the two conditions

$$s_i(t_{i-1})^{-1} t_i \in T_i$$

and

$$t_{i-1}^{-1} t_i \in T_i$$

are equivalent for each $i = 1, \ldots, \ell$, this completes the proof of the part (i).

The part (ii) is an immediate consequence of the projectivity of $\overline{pr}_{0\ell} : \overline{O(s_1, \cdots, s_\ell)} \to \overline{O(w)}$. ∎

Remark (2.4.2). It is not difficult to check by induction on $\ell(w)$ that the diagonalizable group

$$\operatorname{Ker}\left(\prod_{\alpha\in R(w)} \alpha^{V} : \mathbf{G}_{m,k}^{R(w)} \to T_{w}\right)$$

is in fact connected, i.e., a torus, under our hypotheses on G.

The first assertion of (2.3.4) is an immediate consequence of the part (ii) of (2.4.1) and of the following proposition:

PROPOSITION 2.4.3. *For each $w \in W$ with shortest expression $w = s_1 \cdots s_\ell$, the canonical map*

$$R\,\overline{pr}_{0\ell!}(\overline{pr}_{01}^{*}\overline{K(s_1)} \otimes \cdots \otimes \overline{pr}_{\ell-1,\ell}^{*}\overline{K(s_\ell)}) \\ \to R\,\overline{pr}_{0\ell*}(\overline{pr}_{01}^{*}\overline{K(s_1)} \otimes \cdots \otimes \overline{pr}_{\ell-1,\ell}^{*}\overline{K(s_\ell)})$$

is an isomorphism in $D_c^b(\overline{X(w)} \times_{\overline{O(w)}} \overline{O(s_1, \cdots, s_\ell)}, \overline{\mathbf{Q}}_\ell)$.

Proof. The argument is similar to the one used for (2.2.7) and we will take some notations and some consequences of the proof of (2.2.7). We have the graph

$$\Gamma \subset T_w \times_k \mathbf{G}_{m,k}^{R(w)} \subset T_w \times_k \mathbf{A}_k^{R(w)}$$

of $\prod_{\alpha\in R(w)} \alpha^{V}$ and we know that

$$\overline{\mathbf{Q}}_{\ell,\Gamma}[\ell(w)]$$

is an irreducible perverse sheaf on $T_w \times_k \mathbf{A}_k^{R(w)}$.

We have the two Fourier-Deligne transformations relative to T_w

$$\mathcal{F}_!, \mathcal{F}_* : D_c^b(T_w \times_k \mathbf{A}_k^{R(w)}, \overline{\mathbf{Q}}_\ell) \to D_c^b(T_w \times_k \mathbf{A}_k^{R(w)}, \overline{\mathbf{Q}}_\ell)$$

and we know that the canonical map

$$\mathcal{F}_!(\overline{\mathbf{Q}}_{\ell,\Gamma}[\ell(w)]) \to \mathcal{F}_*(\overline{\mathbf{Q}}_{\ell,\Gamma}[\ell(w)])$$

is an isomorphism of irreducible perverse sheaves.

Now, we have the following commutative diagram with cartesian square

$$
\begin{array}{ccccc}
\mathbf{A}_k^{R(w)}\times_k \mathbf{G}_{m,k}^{R(w)} & \xleftarrow{\overline{pr}_{s_1,\cdots,s_\ell}\times id} & \overline{X(x_1,\cdots,s_\ell)}\times_k \mathbf{G}_{m,k}^{R(w)} & \xrightarrow{a} & \overline{X(x_1,\cdots,s_\ell)} \\
{\scriptstyle id\times(\prod_{\alpha\in R(w)}\alpha^\vee)}\downarrow & \square & {\scriptstyle id\times(\prod_{\alpha\in R(w)}\alpha^\vee)}\downarrow & \square & \downarrow{\scriptstyle \overline{pr}'_{0\ell}} \\
\mathbf{A}_k^{R(w)}\times_k T_w & \xleftarrow[\overline{pr}_{s_1,\cdots,s_\ell}\times id]{} & \overline{X(s_1,\cdots,s_\ell)}\times_k T_w & \xrightarrow[b]{} & \overline{X(w)}\times_{\overline{O(w)}}\overline{O(s_1,\cdots,s_\ell)}
\end{array}
$$

where

$$\overline{pr}_{s_1,\cdots,s_\ell}:\overline{X(s_1,\cdots,s_\ell)}\longrightarrow \mathbf{A}_k^\ell=\mathbf{A}_k^{R(w)}$$

is equal to

$$\overline{pr}_{s_1}\times_X\overline{pr}_{s_2}\times_X\cdots\times_X\overline{pr}_{s_\ell},$$

with

$$\overline{pr}_s:\overline{X(s)}=X\times_{Y_s}X\xrightarrow{(,)_s}\mathbf{A}_k^1,$$

and

$$\overline{X(s_1,\cdots,s_\ell)}\times_k\mathbf{G}_{m,k}^{R(w)}\xrightarrow{a}\overline{X(s_1,\cdots,s_\ell)}$$

is the natural action described above and

$$\overline{X(s_1,\cdots,s_\ell)}\times_k T_w\xrightarrow{b}\overline{X(w)}\times_{\overline{O(w)}}\overline{O(s_1,\cdots,s_\ell)}$$

maps $((g_0U,\cdots,g_\ell U),t_w)$ to

$$(g_0U,g_1B,\cdots,g_{\ell-1}B,g_\ell t_wU)$$

(cf. (2.4.1) (i)).

Then it follows easily from the proper and smooth base change theorems ([SGA 4] (XVII, (5.2.6)) and (XVI, (1.1)) that we have a commutative diagram in $D_c^b(\overline{X(s_1,\cdots,s_\ell)}\times_k T_w,\overline{\mathbf{Q}}_\ell)$

$$
\begin{array}{ccc}
b^*R\overline{pr}'_{0\ell!}(\overline{pr}^*_{01}\overline{K(s_1)}\otimes\cdots\otimes\overline{pr}^*_{\ell-1,\ell}\overline{K(s_\ell)}) & \simeq & (\overline{pr}_{s_1,\cdots,s_\ell}\times id)^*\mathcal{F}_!(\overline{\mathbf{Q}}_{\ell,\Gamma}[\ell(w)])(\ell(w)) \\
\downarrow & & \downarrow\wr \\
b^*R\overline{pr}'_{0\ell *}(\overline{pr}^*_{01}\overline{K(s_1)}\otimes\cdots\otimes\overline{pr}^*_{\ell-1,\ell}\overline{K(s_\ell)}) & \simeq & (\overline{pr}_{s_1,\cdots,s_\ell}\times id)^*\mathcal{F}_*(\overline{\mathbf{Q}}_{\ell,\Gamma}[\ell(w)])(\ell(w))
\end{array}
$$

where the two horizontal arrows are isomorphisms and the two vertical ones are canonical.

As $\overline{pr}_{s_1,\cdots,s_\ell} \times id$ and b are smooth with connected geometric fibers, the proposition follows from [B-B-D] (4.2) and we get moreover that the two isomorphic objects of $D_c^b(\overline{X(w)} \times_{\overline{O(w)}} \overline{X(s_1,\cdots,s_\ell)}, \overline{\mathbf{Q}}_\ell)$ of the statement are in fact, up to a shift of dim X, irreducible perverse sheaves. ∎

(2.5) We will now prove the second assertion of (2.3.4) by induction on $\ell(w)$.

Let $w \in W$ and $s \in S$ such that

$$\ell(ws) = \ell(w) + 1.$$

We will assume that (2.3.4) is proved for w. Then, it follows from (2.4.3) that the canonical map

$$R\, p_{13!}(p_{12}^* \overline{K(w)} \otimes p_{23}^* \overline{K(s)}) \\ \rightarrow R\, p_{13*}(p_{12}^* \overline{K(w)} \otimes p_{23}^* \overline{K(s)},$$

where

$$p_{12}, p_{23}, p_{13} : X \times_k X \times_k X \rightarrow X \times_k X$$

are the canonical projections, is an isomorphism in $D_c^b(X \times_k X, \overline{\mathbf{Q}}_\ell)$. These two isomorphic objects of $D_c^b(X \times_k X, \overline{\mathbf{Q}}_\ell)$ are clearly supported by $\overline{X(ws)}$ and it follows from (2.2.10) and (2.3.3)(ii) that they both coincide with $K(ws)$ on $X(ws) \subset \overline{X(ws)}$. So, to finish the proof of (2.3.4) we are reduced to prove that these two isomorphic objects of $D_c^b(X \times_k X, \overline{\mathbf{Q}}_\ell)$, shifted by dim X, are in fact irreducible perverse sheaves on $X \times_k X$.

Let

$$\mathcal{F}_{s!}, \mathcal{F}_{s*} : D_c^b(X \times_k V_s) \rightarrow D_c^b(X \times_k V_s)$$

the two Fourier-Deligne transformations relative to $X \times_k Y_s$ defined by

$$\mathcal{F}_{s!}(-) = R\, q_{13!}(q_{23}^* \mathcal{L}_\psi(\langle , \rangle_s) \otimes q_{12}^*(-))[2](1)$$

and

$$\mathcal{F}_{s*}(-) = R\, q_{13*}(q_{23}^* \mathcal{L}_\psi(\langle , \rangle_s) \otimes q_{12}^*(-))[2](1)$$

where

$$q_{12}, q_{13} : X \times_k V_s \times_{Y_s} V_s \rightarrow X \times_k V_s$$

$$q_{23} : X \times_k V_s \times_{Y_s} V_s \rightarrow V_s \times_{Y_s} V_s$$

are the canonical projections. It is known that the canonical map

$$\mathcal{F}_{s!}(-) \to \mathcal{F}_{s*}(-)$$

is an isomorphism of functors, that these two isomorphic functors are exact in the perverse sense and that they induce an equivalence of abelian categories between the subcategories of perverse sheaves ([La] (1.3.1.1) and (1.3.2.3)).

We have clearly a commutative diagram in $D_c^b(X \times_k X, \overline{\mathbf{Q}}_\ell)$

$$\begin{array}{ccc} R\, p_{13!}(p_{12}^* \overline{K(w)} \otimes p_{23}^* \overline{K(s)}) \simeq (id \times j_s)^* \mathcal{F}_{s!} & (id \times j_s)_! \overline{K(w)} \\ \downarrow \wr & \downarrow \\ R p_{13*}(p_{12}^* \overline{K(w)} \otimes p_{23}^* \overline{K(s)}) \simeq (id \times j_s)^* \mathcal{F}_{s*} & R(id \times j_s)_* \overline{K(w)} \end{array}$$

with $j_s : X \hookrightarrow V_s$ the inclusion (cf. (2.1)), where the two horizontal arrows are isomorphisms and the two vertical ones are canonical. So the second assertion of Theorem 2.3.4 is a consequence of the following lemma:

LEMMA 2.5.1. *Let $\pi : V \to Y$ be as in (1.2), let $Z \overset{i}{\hookrightarrow} V$ be the zero section of this vector bundle and let $X = V - Z \overset{j}{\hookrightarrow} V$. Let A be an irreducible perverse sheaf on X such that the canonical map*

$$j^* \mathcal{F} j_! A \to j^* \mathcal{F} R j_* A$$

is an isomorphism in $D_c^b(X, \overline{\mathbf{Q}}_\ell)$, where $\mathcal{F}$ is the Fourier-Deligne transformation on V relative to Y (cf. (1.2)). Then $j_! A$ and $Rj_* A$ are both isomorphic to $j_{!*} A$ (and in particular are irreducible perverse sheaves on V); $j^* \mathcal{F} j_! A$ and $j^* \mathcal{F} R j_* A$ are isomorphic irreducible perverse sheaves on X.

Proof. It is enough to prove that

$${}^p\mathcal{H}^n(j_! A) \quad \text{and} \quad {}^p\mathcal{H}^n(Rj_* A) \quad (n \in \mathbf{Z}, n \neq 0)$$

and

$$\mathrm{Ker}({}^p\mathcal{H}^0(j_! A) \to {}^p\mathcal{H}^0(Rj_* A))$$
$$\mathrm{Coker}({}^p\mathcal{H}^0(j_! A) \to {}^p\mathcal{H}^0(Rj_* A))$$

are all zero. But these perverse sheaves are clearly supported by Z as their images by $\mathcal{F}$ ($\mathcal{F}$ is exact in the perverse sense; ${}^p\mathcal{H}^n(j_! A) = 0$ if $n > 0$ and ${}^p\mathcal{H}^n(Rj_* A) = 0$ if $n < 0$ ([B-B-D] (4.2.4))) and the lemma follows. ∎

Remark(2.5.2). We have also proved that for each $w \in W$ and each $s \in S$ such that

$$\ell(ws) = \ell(w) + 1.$$

the canonical maps

$$(id \times j_s)_! \overline{K(w)} \to (id \times j_s)_{!*} \overline{K(w)} \to R(id \times j_s)_* \overline{K(w)}$$

are isomorphisms in $D_c^b(X \times_k V_s, \overline{\mathbf{Q}}_\ell)$.

(2.6) For each $w \in W$, we have two functors

$$F_{w,!} = F_{w,\psi,!} = R\,\overline{pr}_!(\overline{K_\psi(w)} \otimes \overline{pr}'^*(-1))$$
$$F_{w,*} = F_{w,\psi,*} = R\,\overline{pr}_*(\overline{K_\psi(w)} \otimes \overline{pr}'^*(-1))$$

from $D_c^b(X, \overline{\mathbf{Q}}_\ell)$ to itself with a canonical morphism of functors

$$can_w = can_{w,\psi} : F_{w,!} \to F_{w,*}$$

between them, where

$$pr, pr' : \overline{X(w)} \to X$$

are induced by the two canonical projections of $X \times_k X$.

PROPOSITION 2.6.1. *For each $w_1, w_2 \in W$, we have canonical morphisms of functors*

$$c_{w_1,w_2,!} = c_{w_1,w_2,\psi,!} : F_{w_1,!}F_{w_2,!} \to F_{w_1w_2,!}$$

and

$$c_{w_1,w_2,*} = c_{w_1,w_2,\psi,*} : F_{w_1w_2,*} \to F_{w_1,*}F_{w_2,*}$$

such that:

(i) *for each $w_1, w_2, w_3 \in W$, we have*

$$c_{w_1,w_2w_3,!} \circ F_{w_1,!}(c_{w_2,w_3,!}) = c_{w_1w_2,w_3,!} \circ c_{w_1,w_2,!}(F_{w_3,!})$$

and

$$F_{w_1,*}(c_{w_2,w_3,*}) \circ c_{w_1,w_2w_3,*} = c_{w_1,w_2,*}(F_{w_3,*}) \circ c_{w_1w_2,w_3,*};$$

(ii) *if*

$$\ell(w_1w_2) = \ell(w_1) + \ell(w_2),$$

then $c_{w_1,w_2,!}$ and $c_{w_1,w_2,*}$ are isomorphisms.

Proof. It follows from (2.3.4) and the proper and smooth base change theorems ([SGA 4] (XVII, (5.2.6)) and (XVI, (1.1)) that, for each $w \in W$ with shortest expression

$$w = s_1 \dots s_\ell ,$$

we have canonical isomorphisms

$$F_{s_1,!} \dots F_{s_\ell,!} \xrightarrow{\sim} F_{w,!}$$

and

$$F_{w,*} \xrightarrow{\sim} F_{s_1,*} \dots F_{s_\ell,*}.$$

Consequently, we are essentially reduced to construct $c_{s,s,!}$ and $c_{s,s,*}$ for each $s \in S$.

But, we have canonical maps in $D_c^b(X \times_{Y_s} X, \overline{\mathbf{Q}}_\ell)$

$$R\, pr_{13!}(pr_{12}^* \overline{K(s)} \otimes pr_{23}^* \overline{K(s)}) \to \Delta_* \overline{\mathbf{Q}}_\ell$$

and

$$\Delta_* \overline{\mathbf{Q}}_\ell \to R\, pr_{13*}(pr_{12}^* \overline{K(s)} \otimes pr_{23}^* \overline{K(s)})$$

where

$$pr_{12}, pr_{23}, pr_{13} : X \times_{Y_s} X \times_{Y_s} X \to X \times_{Y_s} X$$

are the canonical projections and

$$\Delta : X \hookrightarrow X \times_{Y_s} X$$

is the diagonal:

$$\mathcal{L}_\psi(\langle x_1, x_2 \rangle_s + \langle x_2, x_3 \rangle_s)|_{\{x_1 = x_3\}}$$

is canonically isomorphic to $\overline{\mathbf{Q}}_\ell$. So we have canonical morphisms of functors

$$F_{s,!} F_{s,!} \to \mathrm{id}$$

and

$$\mathrm{id} \to F_{s,*} F_{s,*}.$$

■

LEMMA 2.6.2. (i) *If* D *is the duality functor on* $D_c^b(X,\overline{\mathbf{Q}}_\ell)$ *, then, for each* $w \in W$ *, we have canonical isomorphisms of functors*

$$D\,F_{w,\psi,!} \simeq F_{w,\psi^{-1},*}\,D$$
$$D\,F_{w,\psi,*} \simeq F_{w,\psi^{-1},!}\,D;$$

moreover

$$D(can_{w,\psi}) = can_{w,\psi^{-1}}(D)$$

and, for each $w_1, w_2 \in W$

$$D(c_{w_1,w_2,\psi,!}) = c_{w_1 w_2,\psi^{-1},*}(D)$$
$$D(c_{w_1,w_2,\psi,*}) = c_{w_1,w_2,\psi^{-1},!}(D).$$

(ii) *For each* $w \in W$, $(F_{w,!}, F_{w^{-1},*})$ *is a pair of adjoint functors between* $D_c^b(X,\overline{\mathbf{Q}}_\ell)$ *and itself; moreover, if we denote by*

$$a_w = a_{w,\psi} : F_{w,!}F_{w^{-1},*} \to id$$
$$b_{w^{-1}} = b_{w^{-1},\psi} : id \to F_{w^{-1},*}F_{w,!}$$

the adjunction maps, we have

$$D(a_{w,\psi}) = b_{w,\psi^{-1}}(D)$$
$$D(b_{w,\psi}) = a_{w,\psi^{-1}}(D).$$

For each $w_1, w_2 \in W$ *, the maps* $c_{w_1,w_2,!}$ *and* $c_{w_2^{-1},w_1^{-1},*}$ *are exchanged by adjunction. For each* $w \in W$ *, we have also*

$$c_{w,w^{-1},!} = a_w \circ F_{w,!}(can_{w^{-1}})$$

and

$$c_{w,w^{-1},*} = F_{w,*}(can_{w^{-1}}) \circ b_w.$$

(iii) *For each* $w \in W$, $F_{w,!}$ *(resp.* $F_{w,*}$ *) is* t*-exact on the right (resp. on the left).*

Proof. When $w = s \in S$, the lemma is a particular case of (1.3.1). The general case follows from (2.6.1). ■

Taking the ${}^p\mathcal{H}^0$ of the functors $F_{w,!}, F_{w,*}$ and of the morphism of functors can_w, $c_{w_1,w_2,*}$, $c_{w_1,w_2,!}$, we get functors

$$ {}^pF_{w,!} = {}^pF_{w,\psi,!} : Perv(X,\overline{\mathbf{Q}}_\ell) \to Perv(X,\overline{\mathbf{Q}}_\ell) $$
$$ {}^pF_{w,*} = {}^pF_{w,\psi,*} : Perv(X,\overline{\mathbf{Q}}_\ell) \to Perv(X,\overline{\mathbf{Q}}_\ell) $$

and canonical morphisms of functors ${}^pcan_w$, ${}^pc_{w_1,w_2,*}$, ${}^pc_{w_1,w_2,!}$.

Statements similar to proposition (2.6.1) and lemma (2.6.2) holds.

We will consider now families

$$ (A_w, \alpha_{w',w})_{w,w' \in W} $$

where, for each $w \in W$, A_w is an object of $Perv(X,\overline{\mathbf{Q}}_\ell)$ and, for each $w, w' \in W$,

$$ \alpha_{w',w} : {}^pF_{w',!}A_w \to A_{w'w} $$

is a map in $Perv(X,\overline{\mathbf{Q}}_\ell)$ such that the following cocycle condition is satisfied:

(2.6.3). *For each* $w, w_1', w_2' \in W$ *with*

$$ \ell(w_1'w_2') = \ell(w_1') + \ell(w_2'), $$

the following diagram in $Perv(X,\overline{\mathbf{Q}}_\ell)$

$$ \begin{array}{ccccc} {}^pF_{w_1',!}{}^pF_{w_2',!}A_w & \overset{c_{w_1',w_2',!}}{\underset{\sim}{\to}} & {}^pF_{w_1'w_2',!}A_w & \overset{\alpha_{w_1'w_2',w}}{\to} & A_{w_1'w_2'w} \\ & {\scriptstyle {}^pF_{w_1',!}(\alpha_{w_2',w})} \searrow & & \nearrow {\scriptstyle \alpha_{w_1',w_2'w}} & \\ & & {}^pF_{w_1',!}A_{w_2'w} & & \end{array} $$

is commutative.

For each $w, w' \in W$, we have by adjunction a map

$$ \beta_{w',w} : A_{w'w} \to {}^pF_{w'*}A_w $$

in $Perv(X,\mathbf{Q}_\ell)$ ($\beta_{w',w}$ is adjoint to $\alpha_{w'^{-1},w'w}$), i.e.,

$$ \beta_{w',w} = {}^pF_{w',*}(\alpha_{w'^{-1},w'w}) \circ b_{w'}(A_{w'w})); $$

the maps $\beta_{w',w}$ satisfy also a cocycle condition similar to (2.6.3).

DEFINITION 2.6.4. *We will say that a family* $(A_w, \alpha_{w',w})_{w,w' \in W}$ *satisfying (2.6.3) is admissible if, for each* $w, w' \in W$ *, the following diagram in* $Perv(X, \overline{\mathbf{Q}}_\ell)$

$$\begin{array}{ccc} {}^pF_{w',!}A_w & & \\ & \searrow \alpha_{w'w} & \\ {}^pcan_{w'}(A_w) \quad \downarrow & & A_{w'w} \\ & \swarrow \beta_{w'w} & \\ {}^pF_{w',*}A_w & & \end{array}$$

is commutative.

Remark (2.6.5). It follows from (2.6.2) (ii) that for an admissible $(A_w, \alpha_{w',w})_{w,w' \in W}$, the condition (2.6.3) is satisfied for any $w_1', w_2' \in W$.

As in (1.4), we can define a category

$$\mathcal{A} = \mathcal{A}_\psi$$

with objects the admissible families $(A_w, \ \alpha_{w',w})_{w,w' \in W}$ and with maps from $(A_{1,w}, \alpha_{1,w',w})_{w,w' \in W}$ to $(A_{2,w}, \alpha_{2,w',w})_{w,w' \in W}$ the families of maps

$$(u_w)_{w \in W} = (A_{1,w} \xrightarrow{u_w} A_{2,w})_{w \in W}$$

in $Perv(X, \overline{\mathbf{Q}}_\ell)$ such that

$$u_{w'w} \circ \alpha_{1,w',w} = \alpha_{2,w',w} \circ {}^pF_{w',!}(u_w)$$

for each $w, w' \in W$.

Then $\mathcal{A}$ is a $\overline{\mathbf{Q}}_\ell$-linear category in a natural way and we have obvious $\overline{\mathbf{Q}}_\ell$-linear functors

$$\mathcal{F}_{w''} = \mathcal{F}_{w'',\psi} : \mathcal{A} \to \mathcal{A} : (A_w, \alpha_{w',w}) \mapsto (A_{ww''}, \alpha_{w',ww''})$$

$$\mathcal{D} = \mathcal{D}_\psi : \mathcal{A}_\psi \to \mathcal{A}_{\psi^{-1}}, \ (A_w, \alpha_{w',w}) \mapsto (DA_w, D\beta_{w',w})$$

such that

$$\mathcal{F}_{w_2''}\mathcal{F}_{w_1''} \simeq \mathcal{F}_{w_1''w_2''}$$

for each $w_1'', w_2'' \in W$ (we have a right action of W on $\mathcal{A}$) and

$$\mathcal{D}_{\psi^{-1}}\mathcal{D}_\psi \simeq id$$

$$\mathcal{D}_\psi \mathcal{F}_{w'',\psi} \simeq \mathcal{F}_{w'',\psi^{-1}}\mathcal{D}_\psi .$$

LEMMA 2.6.6. *$\mathcal{A}$ is an abelian category and $\mathcal{F}_{w''}$ $(w'' \in W)$ and $\mathcal{D}$ are exact functors.*

Proof. The proof is similar to the one of (1.4.2) and the details are left to the reader. ■

Remark (2.6.7). For $G = SL_{2,k}$ and $(\pi : V \to Y) = (\mathbf{A}_k^2 \to Spec(k))$ the constructions of (2.6) and (1.4) coincide.

On the variety $X = G/U$ we have an algebraic action of G by left translation and an algebraic action of T by right translation and these two actions commute. These two algebraic actions on X induce by functoriality actions of $G(k)$ and $T(k)$ on $Perv(X, \overline{\mathbf{Q}}_\ell)$: for each $g \in G(k)$ and for each $t \in T(k)$ we have exact functors

$$L_g : Perv(X, \overline{\mathbf{Q}}_\ell) \to Perv(X, \overline{\mathbf{Q}}_\ell)$$
$$R_t : Perv(X, \overline{\mathbf{Q}}_\ell) \to Perv(X, \overline{\mathbf{Q}}_\ell)$$

such that

$$L_{g_1 g_2} \simeq L_{g_1} L_{g_2} \quad (\forall g_1, g_2 \in G(k))$$
$$R_{t_1 t_2} \simeq R_{t_2} R_{t_1} \quad (\forall t_1, t_2 \in T(k))$$
$$R_t L_g \simeq L_g R_t \quad (\forall g \in G(k), t \in T(k))$$

$$(L_g = (x \mapsto g^{-1}x)^* \quad \text{and} \quad R_t = (x \mapsto xt^{-1})^*).$$

LEMMA 2.6.8. (i) *For each $g \in G(k)$, we have canonical isomorphisms of functors*

$$L_g {}^pF_{w,!} \simeq {}^pF_{w,!} L_g$$
$$L_g {}^pF_{w,*} \simeq {}^pF_{w,*} L_g$$

$(\forall w \in W)$ *and*

$$L_g D \simeq D L_g.$$

(ii) *For each $t \in T(k)$, we have canonical isomorphisms of functors*

$$R_t {}^pF_{w,!} \simeq {}^pF_{w,!} R_{w^{-1}(t)}$$
$$R_t {}^pF_{w,*} \simeq {}^pF_{w,*} R_{w^{-1}(t)}$$

$(\forall w \in W)$ *and*

$$R_t D \simeq D R_t.$$

Proof. The part (i) follows immediately of the G-equivariance of all the constructions in this chapter (in particular, $\langle , \rangle_s$ on $X \times_{Y_s} X$ is G-equivariant, cf. (2.1)).

To check the part (ii) of the lemma, it is clearly enough to check that for each $s \in S$

$$R_t {}^pF_{s,!} \simeq {}^pF_{s,!} R_{s^{-1}(t)} .$$

But, $s^{-1} = s$ and

$$\langle xt^{-1}, x's(t)^{-1} \rangle_s = \langle x, x' \rangle_s$$

for any $(x, x') \in X \times_{Y_s} X$ ($\pi_s(xt^{-1}) = \pi_s(x's(t)^{-1})$ for any $t \in T$ if $\pi_s(x) = \pi_s(x') : s(t)/t \in T_s$), so the lemma follows. ∎

Now, we can extend L_g and R_t to $\mathcal{A}$ in the following way: for each $g \in G(k)$ and each $t \in T(k)$, we have exact functors

$$\mathcal{L}_g = \mathcal{L}_{g,\psi} : \mathcal{A} \to \mathcal{A} : (A_w, \alpha_{w',w}) \mapsto (L_g A_w, L_g \alpha_{w',w})$$
$$\mathcal{R}_t = \mathcal{R}_{t,\psi} : \mathcal{A} \to \mathcal{A} : (A_w, \alpha_{w',w}) \mapsto (R_{w(t)} A_w, R_{w'w(t)} \alpha_{w',w})$$

such that

$$\mathcal{L}_{g_1 g_2} \simeq \mathcal{L}_{g_1} \mathcal{L}_{g_2} (\forall g_1, g_2 \in G(k))$$
$$\mathcal{R}_{t_1 t_2} \simeq \mathcal{R}_{t_2} \mathcal{R}_{t_1} (\forall t_1, t_2 \in T(k))$$
$$\mathcal{R}_t \mathcal{L}_g \simeq \mathcal{L}_g \mathcal{R}_t (\forall g \in G(k), t \in T(k))$$

and

$$\mathcal{D}_\psi \mathcal{L}_{g,\psi} \simeq \mathcal{L}_{g,\psi^{-1}} \mathcal{D}_\psi$$
$$\mathcal{D}_\psi \mathcal{R}_{t,\psi} \simeq \mathcal{R}_{t,\psi^{-1}} \mathcal{D}_\psi$$

and

$$\mathcal{L}_g \mathcal{F}_{w''} \simeq \mathcal{F}_{w''} \mathcal{L}_g$$
$$\mathcal{R}_t \mathcal{F}_{w''} \simeq \mathcal{F}_{w''} \mathcal{R}_{w''^{-1}(t)}$$

$(\forall w'' \in W)$.

In other words, we have a left action of $G(k)$ on $\mathcal{A}$ and a right action of the semi-direct product $T(k) \times W$ (for the natural action of W on T) on $\mathcal{A}$ and these two actions commute.

PROPOSITION 2.6.9. *The abelian category $\mathcal{A}$ is artinian and noetherian. If $(A_w, \alpha_{w',w})_{w,w'\in W}$ is a simple object of $\mathcal{A}$, then each A_w is either simple, either zero.*

Proof. The first assertion follows immediately from the similar assertion for $Perv(X, \overline{\mathbf{Q}}_\ell)$ ([B-B-D] (4.3.1) (i)).

If $(A_w, \alpha_{w',w})_{w,w'\in W}$ is a simple object of $\mathcal{A}$ and if A_1 has a non trivial quotient

$$A_1 \overset{u_1}{\twoheadrightarrow} \overline{A}_1,$$

then we define for each $w \in W$

$$A_w \overset{u_w}{\twoheadrightarrow} \overline{A}_w$$

as the cokernel of

$$\alpha_{w,1} \circ {}^pF_{w,!}(i) : {}^pF_{w,!}(Ker\ u_1) \to A_w$$

where $i : Ker\ u_1 \hookrightarrow A_1$ is the inclusion; for each $w, w' \in W$, the diagram

$$\begin{array}{ccc} {}^pF_{w',!}{}^pF_{w,!}(Ker\ u_1) & \overset{c_{w',w,!}}{\rightarrow} & {}^pF_{w'w,!}(Ker\ u_1) \\ {\scriptstyle {}^pF_{w',!}(\alpha_{w,1}\circ {}^pF_{w,!}(i))}\downarrow & & \downarrow{\scriptstyle \alpha_{w'w,1}\circ {}^pF_{w'w,!}(i)} \\ {}^pF_{w',!}A_w & \overset{\alpha_{w',w}}{\rightarrow} & A_{w'w} \end{array}$$

is commutative (cf. (2.6.5)), so $\alpha_{w',w}$ induces a map

$$\overline{\alpha}_{w',w} : {}^pF_{w',!}\overline{A}_w \to \overline{A}_{w'w}.$$

It is now easy to check that $(\overline{A}_w, \overline{\alpha}_{w',w})_{w,w'\in W}$ is admissible and that

$$(A_w, \alpha_{w',w})_{w,w'\in W} \overset{(u_w)_{w\in W}}{\rightarrow} (\overline{A}_w, \overline{\alpha}_{w',w})_{w,w'\in W}$$

is a non trivial quotient in $\mathcal{A}$. ∎

(2.7) For each $s \in S$, we can identify the abelian category $Perv(V_s, \overline{\mathbf{Q}}_\ell)$ with the category of admissible

$$(A \underset{\beta}{\overset{\alpha}{-}} A')$$

where $A, A' \in ob\ Perv(X, \overline{\mathbf{Q}}_\ell)$ and

$$\alpha : {}^pF_{s,!}A \to A'$$

$$\beta : A' \to {}^pF_{s,*}A$$

are maps in $Perv(X, \overline{\mathbf{Q}}_\ell)$ (cf. (1.5.4)); moreover, the datum of α (resp. β) is equivalent by adjunction to the datum of a map

$$\beta' : A \to {}^pF_{s,*}A' (\text{resp.}\ \alpha' : {}^pF_{s,!}A' \to A).$$

For each $s \in S$, we have an exact functor

$$\lambda_s^* = \lambda_{s,\psi}^* : \mathcal{A} \to Perv(V_s, \overline{\mathbf{Q}}_\ell)$$

defined by

$$\lambda_s^*((A_w, \alpha_{w',w})_{w,w' \in W}) = \left(A_1 \ \frac{\alpha_{s,1}}{\beta_{s,1}} \ A_s \right)$$

where $\beta_{s,1} : A_s \to {}^pF_{s*}A_1$ is deduced by adjunction of $\alpha_{s,s}$; it is obvious that

$$\mathcal{F}_{V_s}\lambda_s^* \simeq \lambda_s^*\mathcal{F}_s$$

and that

$$D_{V_s}\lambda_{s,\psi}^* \simeq \lambda_{s,\psi^{-1}}^*\mathcal{D}_\psi$$

where we denote by $\mathcal{F}_{V_s}$ and D_{V_s} the Fourier-Deligne transformation and the duality functor on $Perv(V_s, \overline{\mathbf{Q}}_\ell)$, respectively.

LEMMA 2.7.1. *For each $s \in S$, the functor*

$$\lambda_s^* : \mathcal{A} \to Perv(V_s, \overline{\mathbf{Q}}_\ell)$$

admits a left adjoint functor

$$\lambda_{s,!} = \lambda_{s,\psi,!} : Perv(V_s, \overline{\mathbf{Q}}_\ell) \to \mathcal{A}$$

and a right adjoint functor

$$\lambda_{s,*} = \lambda_{s,\psi,*} : Perv(V_s, \overline{\mathbf{Q}}_\ell) \to \mathcal{A}.$$

Moreover

(i) $\lambda_{s,!}$ *is right exact and* $\lambda_{s,*}$ *is left exact;*

(ii) *we have*

$$\mathcal{F}_s\lambda_{s,!} \simeq \lambda_{s,!}\mathcal{F}_{V_s}$$

$$\mathcal{F}_s\lambda_{s,*} \simeq \lambda_{s,*}\mathcal{F}_{V_s}$$

and

$$\mathcal{D}_\psi \lambda_{s,\psi,!} \simeq \lambda_{s,\psi^{-1},*} D_{V_s}$$
$$\mathcal{D}_\psi \lambda_{s,\psi,*} \simeq \lambda_{s,\psi^{-1},!} D_{V_s};$$

(iii) *there exists a canonical morphism of functors*

$$\lambda_{s,!} \to \lambda_{s,*}$$

such that the following diagrams commute

$$\begin{array}{ccc} id & \to & id \\ \uparrow & & \downarrow \\ \lambda_{s,!}\lambda_s^* & \to & \lambda_{s,*}\lambda_s^* \end{array} \quad \text{and} \quad \begin{array}{ccc} id & \to & id \\ \downarrow & & \uparrow \\ \lambda_s^*\lambda_{s!} & \to & \lambda_s^*\lambda_{s,*} \end{array}$$

(the two top horizontal arrows are identities, the two bottom horizontal arrows are induced by $\lambda_{s,!} \to \lambda_{s,*}$ *and the four vertical arrows are adjunction maps) and*

$$\mathcal{F}_s(\lambda_{s,!} \to \lambda_{s,*}) = (\lambda_{s,!} \to \lambda_{s,*})\mathcal{F}_{V_s}$$

and

$$\mathcal{D}_\psi(\lambda_{s,\psi,!} \to \lambda_{s,\psi,*}) = (\lambda_{s,\psi^{-1},!} \to \lambda_{s,\psi^{-1},*})D_{V_s};$$

(iv) *the adjunctions maps*

$$\lambda_s^*\lambda_{s,*} \to id$$
$$id \to \lambda_s^*\lambda_{s,!}$$

are isomorphisms.

Proof. The proof is similar to the proofs of (1.5.1) and (1.5.2); we will give only the formula for $\lambda_{s,!}, \lambda_{s,*}$ and $\lambda_{s,!} \to \lambda_{s,*}$ and leave the details to the reader.

For any $(A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} A) \in ob\, Perv(V_s, \mathbf{Q}_\ell)$, we have functorial maps in $Perv(V, \mathbf{Q}_\ell)$

$${}^pF_{ws,!}{}^pF_{s,!}A \oplus {}^pF_{w,!}{}^pF_{s,!}A' \to {}^pF_{w,!}A \oplus {}^pF_{ws,!}A'$$
$${}^pF_{w,*}A \oplus {}^pF_{ws,*}A' \to {}^pF_{ws,*}{}^pF_{s,*}A \oplus {}^pF_{w,*}{}^pF_{s,*}A'$$

and

$${}^pF_{w,!}A \oplus {}^pF_{ws,!}A' \to {}^pF_{w,*}A \oplus {}^pF_{ws,*}A'$$

given by the following matrices

$$N_! = \begin{pmatrix} c_{ws,s,!}(A) & {}^pF_{w,!}\alpha' \\ -{}^pF_{ws,!}\alpha & -c_{w,s,!}(A') \end{pmatrix}$$

$$N_* = \begin{pmatrix} c_{ws,s,*}(A) & {}^pF_{ws,*}\beta' \\ -{}^pF_{w,*}\beta & -c_{w,s,*}(A') \end{pmatrix}$$

and

$$\Delta = \begin{pmatrix} can_w(A) & 0 \\ 0 & can_{ws}(A') \end{pmatrix}.$$

It is easy to see that the admissibility of $(A \frac{\alpha}{\beta} A')$ implies that the product of matrices

$$N_* \Delta N_!$$

is identically zero, so that Δ admits a functorial canonical factorization

$${}^pF_{w,!}A \oplus {}^pF_{ws,!}A' \twoheadrightarrow Coker(N_!) \to Ker(N_*) \hookrightarrow {}^pF_{w,*}A \oplus {}^pF_{ws,*}A'.$$

We set

$$A_{w,!} = Coker(N_!)$$

$$A_{w,*} = Ker(N_*)$$

and

$$u_w = (Coker(N_!) \to Ker(N_!));$$

we have obvious maps

$$\alpha_{w',w,!} : {}^pF_{w',!}A_{w,!} \to A_{w'w,!}$$

$$\alpha_{w',w,*} : {}^pF_{w',*}A_{w,*} \to A_{w'w,*}$$

induced by $c_{w',w,!}, c_{w',ws,!}$ and $c_{w',w,*}$, $c_{w',ws,*}$. Then

$$\lambda_{s,!}(A \frac{\alpha}{\beta} A') = (A_{w,!}, \alpha_{w',w,!})_{w,w' \in W}$$

$$\lambda_{s,*}(A \frac{\alpha}{\beta} A') = (A_{w,*}, \alpha_{w',w,*})_{w,w' \in W}$$

and

$$(\lambda_{s,!} \to \lambda_{s,*})(A \frac{\alpha}{\beta} A') = (u_w)_{w \in W}.$$

■

3. CONJECTURES

(3.0) We will use the notations and the assumptions of the previous chapters.

Let S be a scheme of finite type over k. For $A_1, A_2 \in ob\, Perv(S, \overline{\mathbf{Q}}_\ell)$, we have two kinds of higher extension groups:

– the Yoneda ones

$$Ext^i_{Perv(S,\overline{\mathbf{Q}}_\ell)}(A_1, A_2) \qquad (i \in \mathbf{Z})$$

(by definition, they are zero for $i < 0$),

– the cohomology groups of the following object of $D^b_c(\overline{\mathbf{Q}}_\ell)$,

$$R\Gamma(S, R\, \underline{Hom}_S(A_1, A_2)).$$

It is a non trivial result of Beilinson ([Be]) that these two kinds of higher extension groups coincide. In particular, the Yoneda higher extension groups are in fact finite dimensional $\overline{\mathbf{Q}}_\ell$-vector spaces and are all zero except perhaps a finite number of them.

If D is the duality functor on $Perv(S, \overline{\mathbf{Q}}_\ell)$, it follows from the result of Beilinson and the duality theorem ([SGA 4] (XVIII, (3.1)) and [SGA $4\frac{1}{2}$] [Th. Finitude] (4.3)) that

$$\text{(3.0.1)} \qquad Ext^i_{Perv(S,\overline{\mathbf{Q}}_\ell)}(A_1, DA_2) = (H^i_c(S, A_1 \otimes A_2))^* \quad (i \in \mathbf{Z})$$

Now, we assume that k is the algebraic closure of a finite field $\mathbf{F}_q$ and that S is defined over $\mathbf{F}_q$; we denote by $Frob_q$ the geometric Frobenius of S relative to $\mathbf{F}_q$ and by S^{Frob_q} or $S(\mathbf{F}_q)$ the finite set of $\mathbf{F}_q$-rational points of S.

We can consider the abelian category

$$Perv_{\mathbf{F}_q}(S, \overline{\mathbf{Q}}_\ell)$$

of pairs

$$(A, \varphi)$$

where $A \in ob\, Perv(S, \overline{\mathbf{Q}}_\ell)$ and

$$\varphi : Frob_q^* A \xrightarrow{\sim} A$$

is an isomorphism in $Perv(S, \overline{\mathbf{Q}}_\ell)$. For each $(A, \varphi) \in ob\, Perv_{\mathbf{F}_q}(S, \overline{\mathbf{Q}}_\ell)$, we have a corresponding function

$$t_{(A,\varphi)} : S^{Frob_q} \to \overline{\mathbf{Q}}_\ell$$

defined by

$$t_{(A,\varphi)}(s) = tr(\varphi_s : A_s \to A_s).$$

If (A_1, φ_1) and (A_2, φ_2) are objects of $Perv_{\mathbf{F}_q}(S, \overline{\mathbf{Q}}_\ell), \varphi_1$ and φ_2 induce an automorphism φ^i of the Yoneda higher extension group

$$Ext^i_{Perv(S,\overline{\mathbf{Q}}_\ell)}(A_1, DA_2) \qquad (i \in \mathbf{Z})$$

and it follows from (3.0.1) and the Grothendieck trace formula ([SGA 5] (XV, §3) or [SGA $4\frac{1}{2}$][Rapport](3.2)) that

$$\begin{aligned}&\sum_{i\in\mathbf{Z}}(-1)^i tr(\varphi^i, Ext^i_{Perv(S,\overline{\mathbf{Q}}_\ell)}(A_1, DA_2))\\ &= \sum_{s\in S^{Frob_q}} t_{(A_1,\varphi_1)}(s)\, t_{(A_2,\varphi_2)}(s).\end{aligned} \tag{3.0.2}$$

In fact, the fomula (3.0.2) defines a symmetric $\mathbf{Z}$-bilinear map

$$K^0(Perv_{\mathbf{F}_q}(S, \overline{\mathbf{Q}}_\ell)) \times K^0(Perv_{\mathbf{F}_q}(S, \overline{\mathbf{Q}}_\ell)) \to \overline{\mathbf{Q}}_\ell \tag{3.0.3}$$

where $K^0(Perv_{\mathbf{F}_q}(S, \overline{\mathbf{Q}}_\ell))$ is the Grothendieck group of the abelian category $Perv_{\mathbf{F}_q}(S, \overline{\mathbf{Q}}_\ell)$.

The group $K^0(Perv_{\mathbf{F}_q}(S, \overline{\mathbf{Q}}_\ell))$ is huge, even for $S = Spec(k)$: we have

$$K^0(Perv_{\mathbf{F}_q}(Spec(k), \overline{\mathbf{Q}}_\ell)) = \mathbf{Z}[\overline{\mathbf{Q}}_\ell^\times].$$

But the pairing (3.0.2) is highly degenerate and we have:

LEMMA 3.0.4. *The quotient of* $K^0(Perv_{\mathbf{F}_q}(S, \overline{\mathbf{Q}}_\ell))$ *by the kernel of the pairing (3.0.3) is isomorphic (as* $\mathbf{Z}$*-module) to the finite dimensional* $\overline{\mathbf{Q}}_\ell$*-vector space* $C(S(\mathbf{F}_q), \overline{\mathbf{Q}}_\ell)$ *of functions on* $S(\mathbf{F}_q)$ *with values in* $\overline{\mathbf{Q}}_\ell$.

Proof. We have a $\mathbf{Z}$-linear map from $K^0(Perv_{\mathbf{F}_q}(S, \overline{\mathbf{Q}}_\ell))$ to $C(S(\mathbf{F}_q), \overline{\mathbf{Q}}_\ell)$ induced by $(A, \varphi) \mapsto t_{(A,\varphi)}$. To prove the lemma, it is clearly enough to show that this map is surjective. But, the delta functions δ_s ($s \in S(\mathbf{F}_q)$) generate $C(S(\mathbf{F}_q), \overline{\mathbf{Q}}_\ell)$ as a $\overline{\mathbf{Q}}_\ell$-vector space and, for each $\lambda \in \overline{\mathbf{Q}}_\ell^\times$, we have

$$\lambda \cdot \delta_s = t_{(A,\varphi)}$$

where $A = \overline{\mathbf{Q}}_{\ell,\{s\}}$ and φ is the multiplication by λ. So the lemma is proved.

More generally, if $\sigma : S \to S$ is an automorphism of finite order which is defined over $\mathbf{F}_q$, we can replace $Frob_q$ by $Frob_q \circ \sigma$ in the previous discussion ($Frob_q \circ \sigma$ is in fact the geometric Frobenius for a new $\mathbf{F}_q$-structure on S).

(3.1) Let $\mathcal{A}$ be the abelian category attached to $X = G/U$ as in (2.6). In fact, $\mathcal{A} = \mathcal{A}_\psi$ depends on the choice of an additive character $\psi : \mathbf{F}_p \hookrightarrow \overline{\mathbf{Q}}_\ell^\times$ and we have a duality functor

$$\mathcal{D} : \mathcal{A}_\psi \to \mathcal{A}_{\psi^{-1}}.$$

If $A_1 \in ob\,\mathcal{A}_{\psi^{-1}}$ and $A_2 \in ob\,\mathcal{A}_\psi$, the Yoneda higher extension groups

$$Ext^i_{\mathcal{A}_{\psi^{-1}}}(A_1, \mathcal{D}A_2)$$

are well defined and we conjecture:

CONJECTURE 3.1.1. *For each $A_1 \in ob\,\mathcal{A}_{\psi^{-1}}$ and each $A_2 \in ob\,\mathcal{A}_\psi$, the Yoneda higher extension groups*

$$Ext^i_{\mathcal{A}_{\psi^{-1}}}(A_1, \mathcal{D}\,A_2) \qquad (i \in \mathbf{Z})$$

are finite dimensional $\overline{\mathbf{Q}}_\ell$-vector spaces and are all zero except perhaps a finite number of them.

Now we assume again that k is the algebraic closure of a finite field $\mathbf{F}_q$ and that G, B are defined over $\mathbf{F}_q$ and T is split over $\mathbf{F}_q$; we denote by $Frob_q$ the geometric Frobenius of $X = G/U$ relative to $\mathbf{F}_q$; $Frob_q$ acts on $\mathcal{A}$ by $Frob_q^*$.

We fix some $w \in W$ and we consider the abelian category

$$\mathcal{A}_{\mathbf{F}_q,w} = \mathcal{A}_{\mathbf{F}_q,w,\psi}$$

of pairs

$$(A, \varphi)$$

where $A \in ob\,\mathcal{A}$ and

$$\varphi : \mathcal{F}_w Frob_q^* A \to A$$

is an isomorphism. If $(A_1,\varphi_1)\in ob\,\mathcal{A}_{\mathbf{F}_q,w,\psi^{-1}}$ and $(A_2,\varphi_2)\in ob\,\mathcal{A}_{\mathbf{F}_q,w,\psi}$, φ_1 and φ_2 induce an automorphism φ^i of the Yoneda higher extension group

$$Ext^i_{\mathcal{A}_{\psi^{-1}}}(A_1,\mathcal{D}A_2)$$

and, if we assume the conjecture (3.1.1), we can define a $\mathbf{Z}$-bilinear map

$$K^0(\mathcal{A}_{\mathbf{F}_q,w,\psi^{-1}})\times K^0(\mathcal{A}_{\mathbf{F}_q,w,\psi})\to\overline{\mathbf{F}}_\ell, \tag{3.0.3}$$

where $K^0(\mathcal{A}_{\mathbf{F}_q,w})$ is the Grothendieck group of the abelian category $\mathcal{A}_{\mathbf{F}_q,w}$, by sending $((A_1,\varphi_1),(A_2,\varphi_2))$ to

$$\sum_{i\in\mathbf{Z}}(-1)^i tr(\varphi^i,Ext^i_{\mathcal{A}_{\psi^{-1}}}(A_1,\mathcal{D}A_2)).$$

This pairing is symmetric up to the replacement of ψ by ψ^{-1}.

CONJECTURE 3.1.3. *The quotient of* $K^0(\mathcal{A}_{\mathbf{F}_q,w,\psi})$ *by the kernel of the pairing (3.1.2) is a finite dimensional* $\overline{\mathbf{Q}}_\ell$*-vector space.*

Let us denote by $E_{\mathbf{F}_q,w,\psi}$ the quotient of $K^0(\mathcal{A}_{\mathbf{F}_q,w,\psi})$ by the kernel of the pairing (3.1.2). It is easy to see that the group $G(\mathbf{F}_q)$ and the twisted finite torus

$$T(\mathbf{F}_q,w)=\{t\in T(k)\,|\,Frob_q(t)=w(t)\}$$

act on $E_{\mathbf{F}_q,w,\psi}$ ($G(k)$ and $T(k)\times W$ act on $\mathcal{A}_\psi$ by $\mathcal{L}_g$, $\mathcal{R}_t$ and $\mathcal{F}_w$ and we have

$$\mathcal{L}_g\,Frob_q^*=Frob_q^*\mathcal{L}_{Frob_q(g)}$$

$$\mathcal{R}_t\,Frob_q^*=Frob_q^*\mathcal{R}_{Frob_q(t)}$$

and

$$\mathcal{F}_wFrob_q^*=Frob_q^*\mathcal{F}_w);$$

these two actions commute. Assuming the conjectures (3.1.1) and (3.1.3) and decomposing the representation $E_{\mathbf{F}_q,w,\psi}$ of the product $G(\mathbf{F}_q)\times T(\mathbf{F}_q,w)$ we expect to get a new construction of the discrete series representation of the finite reductive group $G(\mathbf{F}_q)$.

Remarks (3.2.1). The conjecture (3.1.1) and the case $w=1$ of the conjecture (3.1.3) look much easier than the general case of the conjecture (3.1.3). The problem is to construct by gluing a derived category, equivalent to $D^b(\mathcal{A})$, and an object of $D^b_c(Spec(k),\overline{\mathbf{Q}}_\ell)$ such that the Yoneda higher extension groups are the cohomology groups of this complex.

(3.2.2). We let the reader formulate conjectures analogous to (3.1.1) and (3.1.3) for the category $\mathcal{A}$ constructed in (1.4).

ACKNOWLEDGEMENTS

We wish to thank S. Gelfand and M. Graev for sending us their preprint on the realization of an action of the Weyl group by Fourier transformations in the space of representation of the principal series over a local field. This paper is an attempt to find an algebraic formulation for their construction.

We wish to thank P. Deligne for very useful conversations. The second author would like to thank Harvard University for its hospitality during the preparation of this paper.

We thank L. Schlesinger for her superb typing job.

In the work [G-G] of I. Gelfand and M. Graev it was realized that different series of representations of reductive groups should be considered as «forms» of the principal series. This point of view was made precise for SL_2 in [G-G-PS]. In our paper we propose a way to make precise the statement «Different series of representations of reductive groups are forms of discrete series» for groups over finite fields. It is a special pleasure for us to dedicate this paper to the celebration of the 75th birthday of Professor I.M. Gelfand.

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Manuscript received: December 14, 1988